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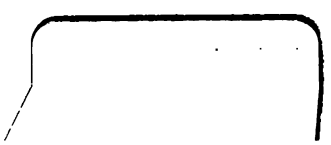

SOLUTIONS

OF

GOODWIN'S COLLECTION

OF

PROBLEMS AND EXAMPLES.

BY THE

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PREFACE.

THE Problems and Examples which I published as a companion to my Course of Mathematics were, with a few exceptions, unaccompanied by solutions; one principal reason for adopting that method was, that it seemed to be a better exercise for a student to attempt the solution of a problem without the assistance which may be derived from inspection of the method to be adopted, or perhaps even from mere knowledge of the result. This opinion I still hold; and I shall be sorry if the present volume be found to impair the usefulness of the other, a result which *must* ensue if its pages be consulted before the student has fully exercised his own powers in attempts at independent solution of the problems.

A volume of solutions of my Collection of Problems and Examples being much demanded, and my own engagements rendering it impossible for me to undertake the work, I entrusted the execution of it to the Rev. W. W. Hutt, Fellow and Sadlerian Lecturer of Caius College. That gentleman kindly undertook the office,

and has prepared the work for publication as rapidly as the pressure of other engagements would allow. Mr Hutt has been assisted in a considerable portion of the work by Mr P. H. Mason, B.A., Scholar of St John's College.

The references throughout the following pages are to the *second edition* of my Course of Mathematics.

H. GOODWIN.

CAMBRIDGE,
May, 1849.

ALGEBRA.

USE OF SYMBOLS.

IN these Examples each letter must be replaced by the number of which it is the representative (Art. 2). The expressions thus obtained must then be reduced by Arithmetical

ERRATA.

Page 21, line 18. The value of x should be the square of that in the text.
— 22. The observation on Ex. 34 also applies to Exs. 24, and 33.
— 49, line 8, *for* ar^{p-q+1} *read* $ar^{p \cdot q-1}$.
— 53, last line, *for* $(p-r)$ *read* $(r-p)$.
— 56, last line but one, *for* 32 *read* 13.
— 134, line 20, *for* W *read* W'.

An additional list of errata will be published, if necessary, on the first of October.

5. If x, y, z be replaced by their given values, the given expression becomes

$$(1 + 3 + 5)\ (1 + 3 - 5)\ (1 + 5 - 3) = 9\,(-1)\,(3) = -27.$$

6. $x - \{y - (z - u)\} = x - \{y - z + u\} = x - y + z - u.$

7. Replacing the letters by their numerical values, we have

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 1 + 4 + 9 + 4 + 6 + 12 = 36,$$

$$\text{also } 6abc = 6 \times 1 \times 2 \times 3 = 36,$$

$$\text{and } a^3 + b^3 + c^3 = 1 + 8 + 27 = 36,$$

each of the given expressions being equal to 36, it follows that they are equal to each other.

Similarly, $a^4 + b^4 + c^4 = 98 = 14(a + 2c)$,

$$a^5 + b^5 = 33 = 11c,$$

$$a^2 - ab + b^2 = 3 = c,$$

and $(a + b)(a + c)(b + c) = 60 = 10abc$.

8. Answers. 182, 64, 48, -16, 7245, 3720.

9. When $x = 3$,

$$x^3 - 3x^2 + 3x - 1 = 27 - 27 + 9 - 1 = 8.$$

10. When $x = \frac{1}{12}$,

$$\sqrt{\tfrac{3}{4} - x} + \sqrt{2x} - \tfrac{3}{2}\sqrt{1 - 4x} = \sqrt{\tfrac{3}{4} - \tfrac{1}{12}} + \sqrt{\tfrac{1}{6}} - \tfrac{3}{2}\sqrt{1 - \tfrac{1}{3}}$$

$$= \sqrt{\tfrac{8}{12}} + \sqrt{\tfrac{1}{6}} - 3\sqrt{\tfrac{1}{4} \cdot \tfrac{2}{3}} = 2\sqrt{\tfrac{1}{6}} + \sqrt{\tfrac{1}{6}} - 3\sqrt{\tfrac{1}{6}} = 0.$$

11. Ans. $(x + y)^{\frac{3}{2}}$, $ax^{\frac{1}{2}} + b^{\frac{3}{2}}x^{\frac{1}{2}} + c^{\frac{3}{2}} \cdot x^{\frac{1}{2}}$, $a^{\frac{1}{2}}$, $a^{\frac{1}{2}} \cdot b^{\frac{1}{2}} \cdot c^{\frac{1}{2}}$.

ADDITION.

1. Ans. $4x^2 + 3ax + 2a^2$.

2. ... $4x^m + 5ax^{m-1} + 4a^2x^{m-2} - 5a^m$.

3. ... $4a + 4b - 3c + d$.

4. ... $6a - d$.

5. ... $4a^3 - a^2b - 2ab^2$.

6. ... $2ab + 2ac - 2ad + 2bd - 2bc + 2cd$.

7. ... $2a^2b - a^2c + a^2d + b^2c + b^2d - abc$.

8. ... $(6a - 4c)x$.

9. ... $2ax$.

10. ... $(2x - 5y)y$.

11. ... $(7a + 3b)x - (a - 6b)y$.

12. ... $4a - 2b - 2c + 2d$.

13. Ans. $3a^3 + a^2b + ac^2 + ab^2 - b^2c$.

14. ... $5a^4 + a^2b^2 + b^4$.

15. ... $a^2b^2c^2 + 4ab^3c^2 + abc^4 + 6a^2b^4 - b^6$.

16. ... $3a^2 + 6b^2$.

17. ... $203a^3 - 157a^2b + 60ab^2 + 9b^3$.

18. ... $175a^4 - 21a^3b + 19a^2b^2 + 20ab^3 - 8b^4$.

SUBTRACTION.

1. Ans. $a + 2b - 2c + 4d$.

2. ... $-a + b + 5c$.

3. ... $ay + 4by + 2ax - 2bx$.

4. ... $2a^2x + 2aby - 3b^2x$.

5. ... $3a^3 + a^2x + 14ax^2 + 5x^3$.

6. ... $8ax + 53ay - 42bx + 2by$.

7. ... $6a^{\frac{1}{2}}b^{\frac{1}{3}} - a^{\frac{3}{2}} + 5b^{\frac{3}{2}}$.

8. ... $3a - 11b$.

9. ... $-(a-d)(x^3-y^3) - (b-c)(x^2y - xy^2)$.

10. ... $(a-b+d)x^2y - (a+b-c)x^3 - (a-b+c)y^3$.

11. ... $4y^3 - 9xy^2$.

12. ... $a + 4b - 3c + 2d$.

13. ... $3a^3 + 4a^2b - 7ab^2$.

14. ... $2a^4 + 3a^3b + a^2b^2 + ab^3$.

15. ... $-10a^4b + a^3b^2 + 2a^2b^3 + 3ab^4 + 2b^5$.

16. ... $2a^4 - 4ab^3 + 6b^4$.

17. ... $-2b^2 - 2c^2 + 2d^2$.

18. ... $22a^3 + 6a^2b - 40ab^2 + 50b^3$.

MULTIPLICATION.

1. Ans. $a^6x^6y^6$.

2. ... $a^{\frac{1}{2}}x^{\frac{1}{2}}$.

3. ... $a^2 - b^2x^2 + 2bcx^3 - c^2x^4$.

4. ... $a^3 + b^3 + c^3 - a^2b + 3a^2c - ab^2 + 3ac^2 - 2abc - b^2c - bc^2$.

5. ... $x^7 + y^7$.

6. ... $2a^7b - 5a^6b^2 - 11a^5b^3 + 5a^4b^4 - 26a^3b^5 + 7a^2b^6 - 12ab^7$.

7. ... $x^6 - 2x^5y + 2x^4y^2 - 4x^3y^3 + 8x^2y^4 + 16xy^5 - 32y^6$.

8. ... $x^4 - 3x^3 + 6x^2 - 5x + 3$.

9. ... $x^{\frac{4}{3}} + 2x - 2x^{\frac{3}{3}} + 7x^{\frac{1}{3}} - 2$.

10. ... $x - x^{\frac{m}{m+n}}.y^{\frac{m-n}{m+n}} + (xy)^{\frac{n}{m+n}} + x^{\frac{2m-n}{m+n}}.y^{\frac{m-n}{m+n}} - (xy)^{\frac{2m-2n}{m+n}} + x^{\frac{m-n}{m+n}}.y^{\frac{m}{m+n}} + (xy)^{\frac{m}{m+n}} - x^{\frac{m-n}{m+n}}.y^{\frac{2m-n}{m+n}} + y$.

11. ... $a^3 + b^3 + c^3 - 3abc$.

12. ... $2a^8b^2 - a^8bc - a^7bc^2 - a^6b^2c^2 + a^7b^2c$.

13. ... $x^3 - 6x^2 + 11x - 6$.

14. The product of two quantities of the forms $a + b$ and $a - b$, where a and b are quantities containing more than one term, may be readily found by the formula

$$(a + b).(a - b) = a^2 - b^2.$$

In the example the product of the first two factors

$$= \{(x^2+1) - 2x\}.\{(x^2+1) + 2x\} = (x^2+1)^2 - 4x^2 = x^4 - 2x^2 + 1,$$

and the product of all three is

$$\therefore = \{(x^4 + 1) - 2x^2\}\ \{(x^4 + 1) + 2x^2\} = (x^4 + 1)^2 - 4x^4$$
$$= x^8 - 2x^4 + 1 = (x^4 - 1)^2.$$

15. Ans. $x^3 - (a+b+c)\,x^2 + (ab+ac+bc)\,x - abc.$

16. ... $a^6 - b^6.$

17. ... $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2 + (abc+abd+acd+bcd)\,x + abcd.$

18. ... $1 - x^{n+1}.$

19. ... $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.$

20. ... $a^{pq} - a^q \cdot b^{(q-1)p} - a^{(p-1)q} \cdot b^p + b^{pq}.$

21. ... $a^3 - 64b^2.$

22. As in Ex. 14, The product of the two factors

$$= \left\{\left(1+\frac{x^2}{3}\right) + \left(\frac{x}{2}+\frac{x^3}{4}\right)\right\} \cdot \left\{\left(1+\frac{x^2}{3}\right) - \left(\frac{x}{2}+\frac{x^3}{4}\right)\right\}$$

$$= \left(1+\frac{x^2}{3}\right)^2 - \left(\frac{x}{2}+\frac{x^3}{4}\right)^2 = 1 + \frac{5}{12}x^2 - \frac{5}{36}x^4 - \frac{x^6}{16}.$$

23. Ans. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b.$

24. ... $182a^3 - 52a^{\frac{5}{2}}b^{\frac{1}{2}} - 119a^2b + 34a^{\frac{1}{2}}b^{\frac{3}{2}} + 42a^{\frac{1}{2}}b^{\frac{5}{2}} - 12b^3.$

DIVISION.

1. Ans. $a^4 - a^3b + a^2b^2 - ab^3 + b^4.$

2. ... $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}.$

3. ... $x^3 + y^3.$

4. ... $x^4 - (p-1)\,x^3 - (p-q-1)\,x^2 - (p-1)\,x + 1.$

5. ... $(x-1)\,a + x^2.$

6. ... $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y.$

7. ... $x^{(p-1)q} + x^{(p-2)q} + x^{(p-3)q} + \ldots\ldots + x^2 + x + 1.$

8. ... $\frac{x^2}{2} - \frac{3x}{4} + 6.$

9. Ans. $x^2 - 3x - 3y$.

10. ... $4a^2 - 6ab - 8b^2$.

11. ... $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

12. ... $1 - 2x + x^2$.

13. ... $a^2 - 2ab + b^2 - c^2$.

14. ... $a^4 + 2a^2b^2 - a^2c^2 + b^4 + b^2c^2$.

15. ... $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

16. ... $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

17. ... $x^{\frac{3}{2}} + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{2}}$.

18. ... $x^{\frac{3}{2}} - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{2}}$.

19. ... $1 + x + x^2 + x^3 + \&c.$

20. ... $1 - 2 \cdot \frac{x}{a} + 2 \cdot \frac{x^2}{a^2} - 2 \cdot \frac{x^3}{a^3} + 2 \cdot \frac{x^4}{a^4} - \&c.$

21. ... $1 - x^2 - x^3 + x^5 + x^6 - \&c.$

22. ... $7a^2 + b^2$.

23. ... $6a^2 - 4ab + 2b^2$.

24. The reasoning in the text is not conclusive because the equation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.$$

is not true, unless the remainder at some stage of the operation be taken into account. On putting $x = 1$, the second side of the equation cannot be said to be equal to

$$1 - 1 + 1 - \&c.$$ '*ad infinitum*,'

because in this expression the fractional remainder is entirely left out of consideration.

GREATEST COMMON MEASURE.

1. $a^2 - b^2 \,)\, a^2 + 2ab + b^2 \,(\, 1$

$$a^2 - b^2$$

$$2ab + 2b^2$$

rejecting the factor $2b$, which is manifestly not a common measure, we have $a + b$ for the new divisor; $a^2 - b^2$ is exactly divisible by $a + b$, which therefore is the G.C.M. required.

2. Ans. $a + b$; the factor b^2 being rejected from the first remainder.

3. ... $a + b$.

4. ... $x - 1$.

5. Multiply the *first* of the given expressions by 4, and divide the result by the *second;* rejecting the factor (y) from the remainder; the new divisor is $3x^2 + xy - 4y^2$, dividing the preceding divisor by this, the remainder is found to be divisible by $(-19y)$; rejecting this factor, the remaining factor $x - y$ is the G.C.M. required.

6. Ans. $12x - 5$.

7. ... $3x + 1$.

8. ... $3x^3 - x + 1$.

9. ... $x - 6$.

10. ... $(x - a)^3$.

11. Dividing the *second* expression by the *first*, the remainder is found to be divisible by $p - q$: rejecting this factor, the new divisor is $x^3 + x^2 - x - 1$; divide the preceding divisor by this, and the remainder is divisible by $p + q - 1$; $x^2 - 1$ the remaining factor is the G.C.M. required.

12. Divide *nine times the second* expression by the *first:* the quotient is $3x - (2a + b)$, and the remainder

$$-(2a^2 - 4ab + 2b^2)\,x + (2a^2 - 4ab + 2b^2)\,a,$$

rejecting the factor $-(2a^2 - 4ab + 2b^2)$, the remaining factor $x - a$ is the G.C.M. required.

[OBS. The second expression is multiplied by *nine* to exclude fractional remainders.]

13. Ans. $x - 2$.

14. The given expressions may be put at once under the forms

$$(3x - y)(3x - 2) \text{ and } (2x^2 - y^2)(3x - 2):$$

$3x - y$ and $2x^2 - y^2$ manifestly have no common measure, $3x - 2$ is therefore the G.C.M. required.

15. Ans. $12x + 5$.

16. Arrange the given expressions according to *descending* powers of y, and divide the *first* by the *second*, the remainder is divisible by $x^4 + x^2$, which is not a common measure, the remaining factor $x^2 - y^2$ is the G.C.M. required.

17. Ans. $a^3 + (a^2 + a)y + y^2$.

18. ... $x^2 - x + 1$.

19. ... $x^2 - 2x + 1$.

20. ... $5x^2 - 3x + 4$. This, and EX. 29, are very tedious on account of the high numbers introduced: great care is requisite in working them out.

21. It will be found that $x + 4$, the G.C.M. of the *first* and *second* expressions, is a measure of the *third*; it is therefore the G.C.M of the three. (Art. 29).

22. Ans. $2x^2$.

23. ... $a + 3x$.

24. Divide the *first* expression by the second; the remainder is divisible by $(a - b)$: reject this, and the new divisor is $x^3 - 2ax^2 + a^2x - b^2(a + b)$, divide the preceding divisor by this, and the remainder is divisible by $a(a + 2b)$, the quotient $x^2 - (a - b)x + b^2$ is the G.C.M. required.

25. Ans. $x - 6$.

26. ... $(x-1)^3$.

27. ... $5x^2 - 1$.

28. Reject the factors n and $(-m)$ of the two expressions, and the G.C.M. of the quotients is found to be $x - y$, which therefore is the G.C.M. required.

29. Ans. $x^3 + x^2 - 5x + 3$.

30. The last term in the *second* expression should be -2; it will then be found that the expressions may be put into the forms

$$(e^{2x} - 1)\,(a^3 + 1) \text{ and } (e^x + 1)\left\{a^2(e^x + 1) - 2\right\}:$$

$e^x + 1$ is therefore the G.C.M. required.

LEAST COMMON MULTIPLE.

1. The G.C.M. of $x^2 - 1$ and $(x-1)^2$ is $(x-1)$, therefore their L.C.M.

$$= \frac{x^2 - 1}{x - 1} \times (x-1)^2 = (x+1)\,(x-1)^2 = x^3 + x^2 - x - 1.$$

2. Ans. $9x^4 - 87x^3 + 312x^2 - 492x + 288$.

3. ... $36x^3 - 3ax^2 - 50a^2x + 24a^3$.

4. The L.C.M. of the *first* and *second* expressions is $4x^2 - 1$, the L.C.M. of this and the *third* or $16x^4 - 8x^3 + 2x - 1$ is the L.C.M. required.

5. Ans. $x^4 + 6x^3 + 3x^2 - 26x - 24$.

6. ... $8(1 - x - x^2 + x^3)$.

7. The given expressions may easily be reduced to the forms

$$(x-1)^3,\ (x-1)^2(x+1),\ (x-1)^3(x+1),\ (x-1)^2(x^2+1).$$

Their L.C.M. therefore

$$= (x-1)^3(x+1)(x^2+1)$$
$$= x^6 - 2x^5 + x^4 - x^2 + 2x - 1.$$

8. The G.C.M. of the two expressions $= x - b$, their L.C.M.

$$= x^5 - (5a + b)\,x^4 - (23a^2 - 5ab)\,x^3 + (117a^3 + 23a^2b)\,x^2 - (90a^4 + 117a^3b)\,x + 90a^4b.$$

9. Ans. $a^4 - 5a^2b^2 + 4b^4$.

10. ... $x^6 - 1$.

11. ... $a^6 - a^5b - a^4b^2 + 2a^3b^3 - a^2b^4 - ab^5 + b^6$.

12. The G.C.M. of the given quantities is

$$a\,(3x^2 - 2ax + a^2),$$

and their L.C.M. is

$$90a^2x^5 + 45a^3x^4 - 10a^4x^3 + 15a^5x^2 + 10a^6x.$$

13. Ans. $12x^4 - 14x^3 - 2x$.

14. ... $x^8 - x^7 + x^5 - 2x^4 + x^3 - x + 1$.

15. ... $x^4 + 2x^3 - x - 2$.

16. ... $18x^4 - 45x^3 + 37x^2 - 19x + 6$.

17. ... $21x^4 - 26x^3 - 55x^2 - 6x - 24$.

18. ... $x^5 - 3x^4 - 19x^3 + 3x^2 + 18x$.

FRACTIONS.

As the results are given in the Section under this head, those Examples only will be noticed which present points likely to cause difficulty to the Student.

4. The first fraction is reducible to $\frac{1}{x}$.

5. If the numerator and denominator of this complex fraction be each multiplied by $1 + ba$, it becomes

$$= \frac{a + ba^2 + b - a}{1 + ba - ab + a^2} = \frac{(a^2 + 1)\,b}{a^2 + 1} = b.$$

7. The L.C.M. of the denominators is

$$(a-b)\,(x+a)\,(x+b).$$

9. The last of the three fractions is reducible to

$$-\frac{x^2}{y\,(x+y)}.$$

10. The sum of the first two fractions

$$= \frac{1}{a-b}\left\{\frac{1}{(a-c)\,(x+a)} - \frac{1}{(b-c)\,(x+b)}\right\}$$

$$= \frac{1}{a-b}\left\{\frac{b^2-bc-a^2+ac-(a-b)\,x}{(a-c)\,(b-c)\,(x+a)\,(x+b)}\right\}$$

$$= \frac{1}{a-b}\left\{\frac{c\,(a-b)-(a^2-b^2)-(a-b)\,x}{(a-c)\,(b-c)\,(x+a)\,(x+b)}\right\}$$

$$= \frac{c-a-b-x}{(a-c)\,(b-c)\,(x+a)\,(x+b)}.$$

The sum of the *three* fractions

$$= \frac{c-a-b-x}{(a-c)\,(b-c)\,(x+a)\,(x+b)} + \frac{1}{(a-c)\,(b-c)\,(x+c)}$$

$$= \frac{c^2-ac-bc+ab}{(a-c)\,(b-c)\,(x+a)\,(x+b)\,(x+c)} = \frac{1}{(x+a)\,(x+b)\,(x+c)}.$$

11. The given expression

$$= \frac{(1+\sqrt{5})\,x-4}{2x^2-x+2-\sqrt{5}x} + \frac{(1-\sqrt{5})\,x-4}{2x^2-x+2+\sqrt{5}x}$$

$$= \frac{2\,(x-4)\,(2x^2-x+2)+2\sqrt{5}x\,.\,\sqrt{5}x}{(2x^2-x+2)^2-5x^2}$$

$$= \frac{x^3-2x^2+3x-4}{x^4-x^3+x^2-x+1}.$$

13. The given expression

$$= \frac{(x+\sqrt{x^2-1})^2-(x-\sqrt{x^2-1})^2}{x^2-(\sqrt{x^2-1})^2}$$

$$= 4x\sqrt{x^2-1}.$$

15. The sum of the first two fractions is found (as in Ex. 10) to be $\dfrac{ab - cx}{(a-c)(b-c)(x-a)(x-b)}$; the sum of this and the remaining fraction gives the result.

16. The sum of the first two fractions

$$= \frac{ab(x+c) - cx(a+b)}{(a-c)(b-c)(x-a)(x-b)}. \quad \text{(Ex. 10).}$$

17. The sum of the first two fractions in this case

$$= \frac{ab(x+c) - cx(a+b+1) + ab + a + b - (x+c)}{(a-c)(b-c)(x-a)(x-b)}.$$

20. This Example will be solved easily if we replace

$a^3 + x^3$ and $a^3 - x^3$ by $(a+x).(a^2 - ax + x^2)$,

and $(a-x)(a^2 + ax + x^2)$,

respectively.

21. The given expression

$$= \frac{(x-5)(x-4)}{x(x-6)} \times \frac{(x-6)(x-7)}{x(x-5)}$$

$$= \frac{(x-4)(x-7)}{x^2} = \frac{x^2 - 11x + 28}{x^2}.$$

23. In Examples of this kind, begin the simplification at the lowest part of the Example: thus

$$1 + x + \frac{x}{1 + x + x^2} = \frac{1 + 3x + 2x^2 + x^3}{1 + x + x^2}$$

$$\therefore 1 + \frac{x}{1 + x + \dfrac{x}{1 + x + x^2}} = 1 + x \times \frac{1 + x + x^2}{1 + 3x + 2x^2 + x^3}$$

$$= \frac{1 + 4x + 3x^2 + x^3}{1 + 3x + 2x^2 + x^3},$$

and the given expression $= \dfrac{x + 3x^2 + 2x^3 + x^4}{1 + 4x + 3x^2 + x^3}$.

INVOLUTION AND EVOLUTION.

1. Ans. $x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 1$; $x^{\frac{4}{3}} - 2x + 3x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$;
$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

2. ... $a^3x^6 + 3a^2bx^5 + 3a(ac + b^2)x^4 + (6abc + b^3)x^3$
$+ 3c(ac + b^2)x^2 + 3bc(b + c)x + c^3$;
$x^{15} - 3x^{10} + 3x^5 - 1$.

3. ... $x^4 - 4x^3 + 6x^2 - 4x + 1$;
$x^2 - 8x^{\frac{3}{2}} + 24x - 32x^{\frac{1}{2}} + 16$;
$16x - 96x^{\frac{3}{4}} + 216x^{\frac{1}{2}} - 216x^{\frac{1}{4}} + 81$.

4. ... $-a^5x^{10}y^5$; $xy^2z^{\frac{5}{3}}$; $a^{10}x^5y^{10}$.

5. ... $a^{\frac{m}{2}}b^{\frac{2m}{3}}x^m$; $a^{\frac{m}{4}}bx^{\frac{m}{3}}$; $a^mx^{2m}y^{3m}$.

8. Expanding, $(1 + x)^2 + (1 + x)^3 + (1 + x)^4$

$$= \begin{cases} 1 + 2x + x^2 \\ + 1 + 3x + 3x^2 + x^3 \\ + 1 + 4x + 6x^2 + 4x^3 + x^4 \end{cases}$$

$$= 3 + 9x + 10x^2 + 4x^3 + x^4.$$

9 and 10. See Art. (47).

11. Ans. $2 + 2x + 9x^2 + 2x^3 + 2x^4$.

12. The given expression

$= \{[1 + (x - x^2)] . [1 - (x - x^2)]\}^2 (1 - x + x^2)$

$= (1 - x^2 + 2x^3 - x^4)^2 (1 - x + x^2)$

$= (1 - 2x^2 + \&c....)(1 - x + x^2)$

$= 1 - x + x^2 - 2x^2 + \&c.,$

the coefficient of x^2 is therefore -1.

13. Ans. $a^m x^m$; $a^{\frac{1}{p}} b^{\frac{2}{q}} c^{\frac{1}{r}}$; $8a^3x$; $9a^{\frac{1}{2}}x^{\frac{1}{2}}$; $5a^{\frac{m}{m+n}} b^{\frac{n}{m+n}}$.

14. ... $a^m x^{p+1}$.

15. ... $x^3 - x + 1$.

16. ... $x - x^{\frac{1}{2}} + 1$.

17. ... $x^2 - 2x + 3$.

18. ... $x^3 - x^2 + 2x + 3$.

19. ... $2xy^2 - 3x^2y + 2x^3$.

20. ... $3x^3 - 2x^2 + x - 1$.

21. ... $5a^3 - 3a^2x + x^2$.

22. ... $x + \frac{a}{3} - \frac{b}{2}$.

23. ... $7x^2 - \frac{x}{5} + 3$.

24. In this and similar Examples, care must be taken to arrange the terms according to ascending or descending powers of some letter involved: the term involving the *highest power of* x *in the denominator* will be the term of *lowest* dimensions in x. To extract the square root of the proposed, arrange it thus:

$$\frac{x^2}{y^2} - \sqrt{2}.\frac{x}{y} + \frac{5}{2} - \sqrt{2}.\frac{y}{x} + \frac{y^2}{x^2}:$$

the square root of this is

$$\frac{x}{y} - \frac{1}{\sqrt{2}} + \frac{y}{x}.$$

25. Ans. $x - 1$.

26. ... $x + 3$.

27. ... $2x^2 + 4x - 3$.

28. ... $x^2 - 2x + 1$.

29. Ans. $\frac{ax}{b} - b$.

30. See Ex. 24. The arrangement of the terms is

$$x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6},$$

the cube root of which is

$$x^2 - 2 + \frac{1}{x^2}.$$

31. The fourth root is most easily found by taking the square root of the square root. The square root of the given expression is $x^4 - 2x^3 + 3x^2 - 2x + 1$; and the square root of this is $x^2 - x + 1$, the required fourth root.

32. Ans. $2x^2 + x - 3$.

33. The first term of the root is manifestly x, subtracting x^5 from the given expression, the first term of the remainder is $-5x^4$, and dividing this by $5x^4$ the quotient is -1: subtracting $(x-1)^5$ from the given expression the remainder is zero. $x - 1$ is therefore the required root. (Art. 52).

34. Ans. $x^2 + x + 1$.

35. This is proved by extracting the cube root of $x^3 + 1$ to three terms.

36. Ans. $x^2 + 2x + 1$.

SIMPLE EQUATIONS.

[As the answers are annexed to the equations and problems, those only will be referred to which are likely to present difficulties.]

7. Multiply each side of this equation by $1 - x$ (the product of the denominators).

8. Multiply each side by 60 (the *least common multiple* of the denominators).

14. Multiply each side by $(4x + b)(2x - a)$, then

$$(6x + a)(2x - a) = (3x - b)(4x + b),$$

$$\text{or } -4ax - a^2 = -bx - b^2,$$

$$\text{or } x(b - 4a) = a^2 - b^2,$$

$$\text{and } \therefore\ x = \frac{a^2 - b^2}{b - 4a}.$$

15. Multiply each side by 120.

22. Transposing so as to have the radical by itself on one side, we have

$$\sqrt{a^2 + bx + x^2} = b - a - x,$$

squaring each side,

$$a^2 + bx + x^2 = b^2 + a^2 + x^2 - 2ab - 2bx + 2ax,$$

$$\text{whence } x = b \cdot \frac{b - 2a}{3b - 2a}.$$

23. Squaring each side, we have

$$4a + x = 4b + 4x + x - 4\sqrt{bx + x^2},$$

transposing, so as to get the radical by itself, and dividing by 4, we get

$$\sqrt{bx + x^2} = b + x - a.$$

Then proceed as in the last example.

24. Transpose one of the radicals and square.

27. $1 + x^3$ is divisible by $1 + x$, and $1 - x^3$ is divisible by $1 - x$, the fractions on the first side can therefore be reduced.

28.
$$\frac{1}{\sqrt{1 - x} + 1} = \frac{\sqrt{1 - x} - 1}{(\sqrt{1 - x} + 1)(\sqrt{1 - x} - 1)}$$

$$= \frac{\sqrt{1 - x} - 1}{1 - x - 1} = -\frac{\sqrt{1 - x} - 1}{x},$$

similarly, the second fraction $= \dfrac{\sqrt{1+x}+1}{x}$, and therefore the proposed equation becomes

$$-\frac{\sqrt{1-x}-1}{x}+\frac{\sqrt{1+x}+1}{x}=\frac{1}{x},$$

$$\therefore \sqrt{1+x}=\sqrt{1-x}-1.$$

29. It is easily found that

$$\frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{b+1}{b-1};$$

$$\therefore \frac{a+x}{a-x}=\frac{(b+1)^2}{(b-1)^2},$$

whence x is found.

31. $\dfrac{x-1}{\sqrt{x}+1}$ is equal to $\sqrt{x}-1$.

32. By transposition, we find

$$x\{.875+.0625-.15\}=1+.2+.375,$$

$$\text{or } x(.7875)=1.575, \quad \therefore x=2.$$

33. Multiplying each side by $\sqrt{x+a}$, and transposing, we have

$$\sqrt{x^2-a^2}=b-x-a.$$

34. Squaring each side,

$$\frac{1}{x}+\frac{1}{a}+\frac{2}{\sqrt{ax}}=\frac{1}{a}+\sqrt{\frac{4}{ax}+\frac{9}{x^2}},$$

subtracting $\dfrac{1}{a}$ from each side, and squaring again

$$\frac{1}{x^2}+\frac{4}{ax}+\frac{4}{x\sqrt{ax}}=\frac{4}{ax}+\frac{9}{x^2},$$

$$\text{or } \frac{8}{x^2}=\frac{4}{x\sqrt{ax}}, \quad \text{or } \frac{2}{x}=\frac{1}{\sqrt{ax}}, \quad \text{or } x=4a.$$

35. We easily find that

$$\frac{\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}} = \frac{1-b}{1+b};$$

$$\therefore \text{ squaring } \frac{a-\sqrt{a^2-ax}}{a} = \left(\frac{1-b}{1+b}\right)^2,$$

$$\text{or } \frac{\sqrt{a^2-ax}}{a} = 1-\left(\frac{1-b}{1+b}\right)^2 = \frac{4b}{(1+b)^2} = \left(\frac{2\sqrt{b}}{1+b}\right)^2,$$

$$\text{squaring } \frac{a^2-ax}{a^2} = \left(\frac{2\sqrt{b}}{1+b}\right)^4,$$

$$\text{whence } x = a\left\{1-\left(\frac{2\sqrt{b}}{1+b}\right)^4\right\}.$$

36. $$\frac{\sqrt{1+x}-1}{\sqrt{1-x}+1} = \frac{(\sqrt{1+x}-1)(\sqrt{1-x}-1)}{(\sqrt{1-x}+1)(\sqrt{1-x}-1)}$$

$$= \frac{\sqrt{1-x^2}-\sqrt{1-x}-\sqrt{1+x}+1}{-x},$$

$$\text{so } \frac{\sqrt{1-x}+1}{\sqrt{1+x}-1} = \frac{\sqrt{1-x^2}+\sqrt{1+x}+\sqrt{1-x}+1}{x},$$

therefore the proposed equation becomes, by substitution,

$$2\sqrt{1-x}+2\sqrt{1+x} = ax,$$

$$\therefore 4(1+x) = a^2x^2+4(1-x)-4ax\sqrt{1-x},$$

$$\text{or } a^2x^2-8x = 4ax\sqrt{1-x},$$

dividing by x and squaring

$$a^4x^2-16a^2x+64 = 16a^2-16a^2x,$$

$$\therefore a^4x^2 = 16(a^2-4),$$

$$\text{or } x = \frac{4}{a^2}\sqrt{a^2-4}.$$

QUADRATIC EQUATIONS.

7. $$x^2 - 21 = \sqrt{x^2 - 9},$$

$$\therefore\ x^4 - 42x^2 + 441 = x^2 - 9,$$

$$x^4 - 43x^2 + \left(\frac{43}{2}\right)^2 = \left(\frac{43}{2}\right)^2 - 450 = \frac{49}{4},$$

$$\therefore\ x^2 = \frac{43 \pm 7}{2} = 25 \text{ or } 18;$$

$$x = \pm 5 \text{ or } \pm 3\sqrt{2}.$$

13. We easily find that

$$\frac{\sqrt{x^2 - a^2}}{x} = \frac{x - a}{x + a},$$

$$\therefore\ \frac{x^2 - a^2}{x^2} = \frac{x^2 - 2ax + a^2}{x^2 + 2ax + a^2};$$

divide each side by $x - a$, then

$$\frac{x + a}{x^2} = \frac{x - a}{x^2 + 2ax + a^2},$$

which gives a quadratic equation.

14. $\dfrac{x - 4}{\sqrt{x} + 2}$ is equal to $\sqrt{x} - 2$.

$$\therefore\ \sqrt{x} - 2 = x - 8,$$

$$x - \sqrt{x} + \frac{1}{4} = \frac{25}{4},$$

$$\sqrt{x} = \frac{1 \pm 5}{2} = 3 \text{ or } -2, \quad \therefore\ x = 9 \text{ or } 4.$$

15. The equation should have been

$$\frac{a + x}{\sqrt{a} + \sqrt{a + x}} = \frac{a - x}{\sqrt{a} - \sqrt{a + x}}:$$

this gives, after multiplication and transposition,

$$2a\sqrt{a+x} = 2x\sqrt{a},$$

whence $a^2 + ax = x^2$,

which gives $x = \frac{a}{2}(1 \pm \sqrt{5})$.

16. Squaring each side, we have

$$x^2 + \tfrac{1}{2}\sqrt{x^2 + 1664} = x^2 + 2x + 1;$$

$$\therefore\ x^2 + 1664 = (4x + 2)^2,$$

whence x may be found.

17. Transposing and squaring, we have

$$(x^2 - 2cx + c^2)\,ab = (a - b)^2 cx;$$

$$\therefore\ x^2 - \frac{a^2 + b^2}{ab}\,cx = -c^2,$$

whence x may be found.

18. Multiply each side by $6x(x + 1)$, and the equation is reduced to

$$x^2 + x = 6.$$

21. As in Ex. 28, p. 16, we have

$$\frac{\sqrt{1+x}}{1+\sqrt{1+x}} = -\frac{\sqrt{1+x}\,\{1-\sqrt{1+x}\}}{x},$$

and $$\frac{\sqrt{1-x}}{1-\sqrt{1-x}} = \frac{\sqrt{1-x}\,\{1+\sqrt{1-x}\}}{x},$$

whence $1 + x - \sqrt{1+x} = 1 - x + \sqrt{1-x}$,

or $2x = \sqrt{1+x} + \sqrt{1-x}$,

whence x may be found.

24. Multiply each side by 2, and add $6 - 3x$ to each side, then

$$2x^2 - 5x + 6 + 10\sqrt{2x^2 - 5x + 6} = 39.$$

Let $2x^2 - 5x + 6 = z^2$, then

$$z^2 + 10z = 39;$$

this equation gives $z = 3$ or -13, and $\therefore z^2 = 9$ or 169,

$2x^2 - 5x + 6 = 9$ gives $x = 3$ or $-\frac{1}{2}$,

..... $= 169$ ... $x = \dfrac{5 \pm \sqrt{1329}}{4}$.

28. Multiply each side by $\sqrt{x}(1 + 5\sqrt{x})$, then

$$15x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + 2 + 10x^{\frac{1}{2}} = 3x + 15x^{\frac{3}{2}},$$

$$\text{or } x - \tfrac{5}{3}x^{\frac{1}{2}} = \tfrac{2}{3},$$

whence $x^{\frac{1}{2}} = 2$ or $-\frac{1}{3}$, and $\therefore x = 4$ or $\frac{1}{9}$.

29. Multiply each side by $x^{\frac{3}{2}}$, then

$$x^3 - x^{\frac{3}{2}} = 56,$$

whence $x^{\frac{3}{2}} = 8$ or -7, and $\therefore x = 4$ or $\sqrt[3]{49}$.

30. Divide each side by $x^{\frac{1}{m}}$, then

$$a^2b^2x^{\frac{1}{n}-\frac{1}{m}} - 4a^{\frac{1}{2}}b^{\frac{1}{2}}x^{\frac{1}{2n}-\frac{1}{2m}} = (a-b)^2.$$

Let $x^{\frac{1}{n}-\frac{1}{m}} = z^2$, then

$$a^2b^2z^2 - 4a^{\frac{1}{2}}b^{\frac{1}{2}}z = (a-b)^2,$$

which gives $z = \left(\dfrac{1}{\sqrt{b}} \pm \dfrac{1}{\sqrt{a}}\right)^2$,

$$\therefore x = z^{\frac{mn}{m-n}} = \left(\frac{1}{\sqrt{b}} \pm \frac{1}{\sqrt{a}}\right)^{\frac{2mn}{m-n}}.$$

32. This equation may be written

$$x^{-3} + x^{-\frac{3}{2}} = 2,$$

whence $x^{-\frac{3}{2}} = 1$ or -2, and $\therefore x = 1$ or $\sqrt[3]{\frac{1}{4}}$.

33. Adding $9(1 + x)$ to each side, we have

$$4x^2 + 12x\sqrt{1 + x} + 9(1 + x) = 36(1 + x),$$

extracting the square root of each side,

$$2x + 3\sqrt{1+x} = \pm 6\sqrt{1+x}.$$

Taking the upper sign we have $x = 3$ or $-\frac{3}{4}$.

..... lower ..... $x = \frac{9}{8}\{9 \pm \sqrt{97}\}$.

34. This equation may be reduced to the form

$$x - 4 = \frac{x^2}{(1-\sqrt{1+x})^2},$$

$$\therefore (x-4)\{1 - 2\sqrt{1+x} + 1 + x\} = x^2,$$

$$\text{or } (4-x)\sqrt{1+x} = 4 + x \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\therefore (4-x)^2(1+x) = (4+x)^2 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

whence $x^3 = 8x^2$,

and therefore $x = 0$ or 8.

[Obs. $x = 8$ does not satisfy the proposed equation: the reason of this is that although all the values of x which satisfy (i) necessarily satisfy (ii), yet the values of x deduced from (ii) do *not* necessarily satisfy (i): for, in extracting the square root of each side of (ii), we might take the radical $\sqrt{1+x}$ with either a positive or negative sign. The root $x = 8$ corresponds to this negative sign, and is a solution of the equation

$$1 + \frac{x}{2} - \frac{x^2}{(1+\sqrt{1+x})^2} = 3.]$$

SIMULTANEOUS EQUATIONS.

3. Square each side of the second equation, and subtract the first from the result, we find $xy = \frac{b^2 - a^2}{2}$. Then proceed as in Art. 75. Ex. 3.

4. Cube the second equation and subtract the first from the result,

$$3xy(x+y) = 18, \text{ or since } x + y = 3, \quad xy = 2.$$

Proceed as in the last.

5. From the first equation subtract the fourth power of the second, the result is

$$4x^3y - 6x^2y^2 + 4xy^3 = 256,$$

$$\text{or } xy\{2x^2 - 3xy + 2y^2\} = 128, \quad \ldots\ldots\ldots\ldots \text{(i)}$$

also squaring the second equation, and multiplying the result by $2xy$, we have

$$xy\{2x^2 - 4xy + 2y^2\} = 8xy \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Subtracting (i) from (ii), $x^2y^2 = 128 - 8xy$,

whence we find $xy = 8$ or -16;

knowing xy and $x + y$ we can proceed as in the last Example.

8. Adding the given equations together, we find

$$\sqrt{y} = \tfrac{5}{2}\sqrt{a - x},$$

$$\therefore \sqrt{a - x} = \tfrac{2}{5}\sqrt{y};$$

$$\text{also } \sqrt{y - x} = \tfrac{3}{2}\sqrt{a - x},$$

$$\text{whence } 25a - 25x = 4y,$$

$$\text{and } 4y - 4x = 9a - 9x,$$

from which equations x and y may be determined.

10. Extract the n^{th} root of the first equation and the m^{th} of the second, and divide the first result by the second; then we have

$$\left\{\left(\frac{x}{a}\right)^{\frac{m}{n}} \cdot \frac{1}{b}\right\} \div \left\{\left(\frac{x}{b}\right)^{\frac{n}{m}} \cdot \frac{1}{a}\right\} = c^{\frac{1}{n}} \div d^{\frac{1}{m}},$$

$$\text{or } x^{\frac{m}{n} - \frac{n}{m}} \cdot b^{\frac{n}{m} - 1} = c^{\frac{1}{n}} \cdot d^{-\frac{1}{m}} \cdot a^{\frac{m}{n} - 1},$$

$$\text{or } x^{\frac{m^2 - n^2}{mn}} = c^{\frac{1}{n}} \cdot d^{-\frac{1}{m}} \cdot a^{\frac{m-n}{n}} \cdot b^{\frac{m-n}{m}},$$

$$\text{and } \therefore x = c^{\frac{m}{m^2 - n^2}} \cdot d^{\frac{n}{m^2 - m^2}} \cdot a^{\frac{m}{m+n}} \cdot b^{\frac{n}{m+n}},$$

y may be found by interchanging m with n, and a with b.

11. Multiply the second equation by 20, and add the result to the first: this gives $70x = 1.4$ or $x = .02$; whence $y = 2.9$.

12. The first equation may be written

$$c(ay + bx) = xy(x + y),$$

whence $c(ay + bx - xy) = xy(x + y - c)$

dividing the second equation by this, we have

$$\frac{xy}{c} = \frac{abc}{xy},$$

$$\text{or } xy = \pm c\sqrt{ab},$$

this value of xy being substituted in the given equations, they are easily solved.

13. Subtracting the second equation from c times the first, and the third from c times the second; we obtain the equations

$$(c - a)x + (c - b)y = Ac \quad \text{..... (i)}$$

$$a(c - a)x + b(c - b)y = 0 \quad \text{.....(ii)}$$

Subtracting (ii) from b times (i), we have

$$(b - a)(c - a)x = Abc,$$

$$\text{or } \left(1 - \frac{a}{b}\right)\left(1 - \frac{a}{c}\right)x = A,$$

whence x is found: the values of y and z may then be written down by Art. 75.

14. This Example is solved by the same process as Ex. 5. Art. 75.

15. Assume $y = mx$ (Art. 76), then the given equations give

$$x^2(1 - m) = 2 \text{ and } x^2(2 + m^2) = 9;$$

$$\therefore 9(1 - m) = 2(2 + m^2),$$

this equation gives $m = \frac{1}{2}$ or -5;

$$\therefore x^2 = \frac{2}{1 - m} = 4 \text{ or } \frac{1}{3},$$

$$\text{or } x = \pm \frac{1}{\sqrt{3}} \text{ or } \pm 2,$$

$$\text{and } \therefore y = \mp \frac{5}{\sqrt{3}} \text{ or } \pm 1.$$

16. Extracting the square root of the first equation

$$\sqrt{xy} = \pm 35,$$

$$\text{i. e. } \sqrt{x} \cdot \sqrt{y} = \pm 35,$$

the sum and the product of $\sqrt{x}$ and $\sqrt{y}$ being known, the process is the same as in Ex. 3. Art. 75.

17. Substituting in the second and third equations the value of y deduced from the first, we have

$$x + \sqrt{xz} + z = 14 \ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

$$\text{and } x^2 + xz + z^2 = 84 \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

squaring (α) and subtracting (β) from the result; we obtain

$$2\sqrt{xz}\,(x + \sqrt{xz} + z) = 112,$$

and therefore by (α), $\sqrt{xz} = 4$ or $y = 4$.

The third of the given equations now gives

$$x^2 + z^2 = 68,$$

from which and from $xz = 16$, x and z are easily found.

18. Adding the first equation to twice the second, we find

$$(x + y)^2 + (x + y) = 30,$$

solving this equation with respect to $x + y$, we find

$$x + y = 5 \text{ or } -6,$$

this equation and $xy = 6$, will give x and y.

19. See Equations (α) and (β) in Ex. 17.

20. Adding 2 to each side of the first equation, we have

$$\left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right) = \frac{35}{4},$$

solving this equation with respect to $\frac{x}{y} + \frac{y}{x}$, we find

$$\frac{x}{y} + \frac{y}{x} = \frac{5}{2} \text{ or } -\frac{1}{2} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Taking the former solution,

$$x^2 + y^2 = \frac{5}{2}xy,$$

and from the second given equation

$$x^2 + y^2 = 4 + 2xy,$$

$$\therefore\ 5xy = 8 + 4xy,$$

$$\text{or } xy = 8;$$

this and the second given equation give $x = 4$ or -2, and $y = 2$ or -4.

Taking the second solution of (1), we find

$$x = 1 \pm \sqrt{\tfrac{3}{11}}, \text{ and } y = 1 \mp \sqrt{\tfrac{3}{11}}.$$

21. Dividing the given equations by ayz, bxz, cxy respectively, we have

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{a},$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{b},$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{c},$$

adding the second and third of these equations together and subtracting the first from the result, we find

$$\frac{2}{x} = \frac{1}{b} + \frac{1}{c} - \frac{1}{a},$$

$$\text{whence } x = \frac{2}{\frac{1}{b} + \frac{1}{c} - \frac{1}{a}}.$$

[Obs. In these equations $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ are made the unknown quantities.]

22. Dividing the *third* equation by the *fourth*, we have $\frac{y}{x} = \frac{1540}{660} = \frac{7}{3}$ or $y = \frac{7}{3}x$: dividing the *third* by the *second*, we have $z = \frac{11}{3}x$: substituting these values of y and z in the *first* equation, we find $\frac{77}{9}x^3 = 231$, or $x^3 = 27$, or $x = 3$.

23. Eliminating x between the *first* and *third* equations, and w between the *second* and *fourth*; the two resulting equations and the last of the given equations will determine y, z and u.

24. Multiplying the *first* and *second* equations together, we obtain $x^2 = 9$ or $x = 3$. Dividing the *third* by the *second*, and the *first* by the *third*, we obtain y and z.

26. Dividing the given equations by xy, xz and yz respectively; we obtain three simple equations in $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$. See Ex. 21. on the preceding page.

27. Put $y = mx$, then the given equations become

$$\sqrt{x}\{\sqrt{a} + \sqrt{mb}\} = a + b,$$

$$\text{and } x(1 + m) = 2(a + b),$$

$$\text{whence } (1 + m)(a + b) = 2(\sqrt{a} + \sqrt{mb})^2,$$

$$\text{or } m(a - b) - 4\sqrt{ab}\sqrt{m} = a - b,$$

$$\text{which gives } m = \left(\frac{\sqrt{a} \mp \sqrt{b}}{\sqrt{a} \pm \sqrt{b}}\right)^2.$$

$$\text{Hence } x = \frac{2(a + b)}{1 + m} = (\sqrt{a} \pm \sqrt{b})^2,$$

$$\text{and } y = mx = (\sqrt{a} \mp \sqrt{b})^2.$$

28. Dividing the *second* equation by the square of the *third*, we obtain $\frac{y}{4} = \frac{2}{x^3y^2}$, or $y = \frac{2}{x}$, therefore by the *second* equation $z = 3x$, substituting these values of y and z in the first equation, we find $x = 1$: hence $y = 2$ and $z = 3$.

PROBLEMS PRODUCING EQUATIONS.

5. Let $x =$ a daughter's portion, then $2x$ is a younger son's portion, and $3x$ is the eldest son's portion: therefore by the problem

$$4(x) + 3(2x) + 3x = \text{the whole property} = 26000.$$

Hence $x = 2000$, $2x = 4000$, and $3x = 6000$.

6. Let x be the wages of each in shillings; then, $x + 15$ is to be three times $x - 9$, or, $x + 15 = 3(x - 9)$, whence $x = 21$.

9. Let $x + 10$, $x - 5$, and x be the number of gallons conveyed in a minute by the first, second, and third pipes respectively,

$$\therefore x + 10 + x - 5 + x, \text{ or } 3x + 5$$

is the number of gallons conveyed by them all in *one* minute: but by the question they convey 820 gallons in twenty minutes, and therefore, 41 gallons in *one* minute,

$$\therefore 3x + 5 = 41, \text{ whence } x = 12.$$

10. Let $4x$ and $9x$ be the sums possessed by A and B at first, then $4x + 10$ and $9x - 10$ are the sums they had after A won £10: therefore by the question

$$4x + 10 = 9x - 10, \text{ or } x = 4,$$

whence £16, and £36 are the sums required.

11. The couriers travel $\frac{13}{2}$ and $\frac{26}{3}$ miles an hour respectively; when the second passes through the place, the first

has travelled $\frac{13}{2} \times 12$ or 78 miles: suppose the second to overtake the first in x hours, then

$$\frac{26}{3}x = 78 + \frac{13}{2}x,$$

whence $x = 36$, and $\frac{26}{3} \cdot x = 312$ the distance gone over.

12. Let x be the rate of the tide (in miles) in the middle of the river,

$\therefore \frac{3x}{5}$ = its rate at the sides,

y = the number of miles the waterman can row, in still water, in one hour;

$\therefore y + x$ = his velocity down the stream,

$y - \frac{3x}{5}$ = ... up .....,

and $\frac{18}{y+x}$ and $\frac{18}{y - \frac{3x}{5}}$ are the times with and against the stream,

$$\therefore \frac{18}{y+x} = \tfrac{3}{2}, \text{ and } \frac{18}{y - \frac{3x}{5}} = \tfrac{9}{4}, \text{ whence } x' = 2\tfrac{1}{2}.$$

13. In any time, the hour hand passes over $\frac{1}{12}$th the space passed over by the minute hand. If x therefore be the number of minutes past 4 at which the hands are together, i. e. the number of minute divisions passed over by the minute hand since 4 o'clock, $\frac{x}{12}$ is the number passed over by the hour hand: and at 4, the hour hand was 20′ before the minute hand,

$$\therefore x = \frac{x}{12} + 20, \text{ or } x = 21'\tfrac{9}{11}.$$

14. Let x = the required distance in miles;

$\therefore \frac{x}{2}$ = the rate of sailing with the fair wind,

$\frac{x}{2} - 6 =$ ..... on starting from Calais,

$\frac{x}{2} - 4 =$ ..... when the wind changed,

$\therefore \frac{\frac{x}{2}}{\frac{x}{2} - 6}$ = the time in which it came half-way over,

$\frac{\frac{x}{2}}{\frac{x}{2} - 4} =$ ..... completed the passage,

$\therefore \frac{x}{2}\left(\frac{1}{\frac{x}{2} - 6} + \frac{1}{\frac{x}{2} - 4}\right)$ = the whole time of passage

$$= \frac{6}{7} \frac{x}{\frac{x}{2} - 6}.$$

Hence, $x = 22$ miles.

15. Let x = the number of guineas,

$\therefore x - 9$ = the number of standard guineas of the same aggregate weight,

and $\frac{4}{15}(x - 9)$ = the weight of the x guineas, in ounces,

hence, $\frac{4}{15}\left(1 - \frac{9}{x}\right) =$ ..... each guinea .....

$\therefore \frac{4}{15}\left(1 - \frac{9}{x}\right)\left(\frac{x}{2} + \frac{9}{2}\right)$ = the weight of the part

$$= \frac{4}{15}\left(\frac{x}{2} + \frac{9}{2}\right) - \frac{3}{2}.$$

Hence, $x = 36$, the required number.

16. Let x and y be A's and B's money at first, in shillings; then $x + \frac{x}{2}$, and $y - \frac{x}{2}$ are the sums they had after the first set of games,

$$\therefore\ x + \frac{x}{2} + 15 = 3\left(y - \frac{x}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also $x + \frac{x}{2} - 10$, and $y - \frac{x}{2} + 10$ are the sums they had after the second set of games,

$$\therefore\ 2\left(x + \frac{x}{2} - 10\right) = y - \frac{x}{2} + 10 \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

(i) and (ii) determine x and y.

17. If $\frac{x}{y}$ be the required fraction, we have for the determination of x and y, the two equations

$$\frac{2x}{y+7} = \frac{2}{3}, \text{ and } \frac{x+2}{2y} = \frac{3}{5}.$$

18. Let the casks hold x and y gallons respectively, then by the former part of the question

$$x - 15 : y - 11 :: 8 : 3, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and by the latter part

$$\frac{x}{2} + 10 : \frac{y}{2} + 10 :: 9 : 5, \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

(i) and (ii) determine x and y.

19. Let w be the work,

x and y the number of days in which A and B can respectively do it,

then $\frac{w}{x}$ and $\frac{w}{y}$ are the portions done by them in one day.

$$\therefore\ 16\left(\frac{w}{x} + \frac{w}{y}\right) = \text{the work done by the two in 16 days,}$$

$$= w,$$

$$\text{or}\quad \frac{16}{x} + \frac{16}{y} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also from the latter part of the question, if A work 4 days, and B 40 days, the work will be done;

$$\therefore \frac{4w}{x} + \frac{40w}{y} = w,$$

$$\text{or } \frac{4}{x} + \frac{40}{y} = 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

(i) and (ii) determine x and y.

20. Let a be the number of gallons the cistern will hold,

x the number of hours in which one of the equal cocks will fill it,

y the number of hours in which the third cock will fill it,

$\therefore \frac{2a}{x} + \frac{a}{y}$ = the quantity admitted by the three cocks in one hour,

$$\text{and } \therefore 4\left(\frac{2a}{x} + \frac{a}{y}\right) = \frac{5}{12}a,$$

$$\text{whence } \frac{8}{x} + \frac{4}{y} = \frac{5}{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

From the second hypothesis, we find in the same way

$$\frac{32}{3}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{7}{9} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

(i) and (ii) determine x and y.

24. Let x = the number of sheep in the first flock,

$\therefore x + 18 =$ ..... second ...

Hence $\frac{x+18}{x}$ is the price (in pounds) of a sheep in the first flock,

and $\frac{x}{x+18}$ is the price (in pounds) of a sheep in the second flock,

$$\therefore \frac{6(x+18)}{x} : \frac{7x}{x+18} :: 7 : 6,$$

whence x, and therefore $x + 18$ may be found.

25. Let x and y be the digits from right to left; the number is $10y + x$; and therefore by the question

$$(10y + x)\, y = 46,$$

and $$(x + y)\, y = 10,$$

whence x and y can be determined.

26. Let x and y be the numbers, x being $> y$,

then $xy^3 : x^3y :: 4 : 9$, ... (i)

and $x^3 + y^3 = 35$.....(ii)

From (i), $y^2 : x^2 :: 4 : 9$, or $y = \frac{2x}{3}$,

and therefore by (ii), $x^3 + \frac{8x^3}{27} = 35$,

whence $x = 3$, and $\therefore\ y = 2$.

27. Let x = the number of miles travelled by B per diem.

$\therefore\ x + 8 =$ ..... A .....

and $\frac{x}{2}$ being the number of days they travelled,

$\frac{x^2}{2}$ = the distance B travelled,

$\frac{x(x+8)}{2} =$ ..... A .....

$$\therefore \frac{x^2}{2} + \frac{x^2 + 8x}{2} = 320,$$

whence x may be found.

28. Let x be the number of miles he travelled per hour,

then $\frac{105}{x}$ = the number of hours the journey took.

If he had travelled two miles an hour slower, he would have been $\frac{105}{x-2}$ hours on his journey; hence

$$\frac{105}{x-2} = \frac{105}{x} + 6,$$

whence x may be determined.

29. Let x = A's subscription,

$\therefore\ x + 15 = B$'s .....

Also the shares of the profits received by A, B and C must be respectively proportional to the products of the number of pounds they invested, and the number of months they were engaged in the speculation: hence, if y be A's share of the profits, $\frac{x+15}{x}y$ will be B's share, and $\frac{50 \times 12}{16x}y$ will be C's share; therefore by the question, we have for the determination of x and y,

$$y + \frac{x+15}{x}y + \frac{75}{2x}y = 159, \quad \text{..... (i)}$$

$$\text{and } x + y = 88 \text{.....(ii)}$$

Substituting $88 - x$ for y in (i) and reducing, we get

$$4x^2 + 71x = 9240,$$

whence x may be found.

30. Let x, y, z be the numbers,

$\therefore\ x - y$ and $y - z$ are their differences,

and $x - 2y + z$ the difference of their differences,

$$\therefore\ x - 2y + z = 5, \quad \text{..... (i)}$$

$$\text{also } x + y + z = 20, \quad \text{..... (ii)}$$

$$\text{and } xyz = 130 \text{.....(iii)}$$

Subtracting (i) from (ii), $3y = 15$ or $y = 5$:

(ii) and (iii) then give $x + z = 15$ and $xz = 26$, whence x and z can be found.

31. Let x = the number of yards of the better,

y = ... worse,

$\therefore\ x + y$ = the price (in shillings) of a yard of the better,

$x - y$ = ... worse.

Hence by the problem,

$$x(x+y) + y(x-y) = 158, \quad \text{.....(i)}$$

$$\text{and } x(x+y) : y(x-y) :: 72 : 7 \text{.....(ii)}$$

Putting $y = mx$ in (i) and (ii), we have from (ii),

$$72m^2 - 65m = -7,$$

whence $m = \frac{7}{9}$, and then by (i), $x = 9$, and $\therefore\ y = 7$.

32. Let $1 : x$ be the proportion of gold to silver in the first mixture,

$1 : y$ be the proportion of gold to silver in the second mixture.

Also, let s be the value of an ounce of silver,

and $\therefore$ $13s$ ..... gold:

then, supposing a coin of the first metal to weigh a ounces, it will contain $\frac{a}{1+x}$ ounces of gold, and $\frac{ax}{1+x}$ of silver: the value of the gold will be $\frac{13as}{1+x}$, and of the silver $\frac{axs}{1+x}$:

$$\text{and therefore the value of the coin} = \frac{13+x}{1+x}.as.$$

Similarly,

$$\left.\begin{array}{c}\text{The value of a coin of the second}\\ \text{material weighing } a \text{ ounces}\end{array}\right\} = \frac{13+y}{1+y}.as,$$

$\therefore$ by the question, $\frac{13+x}{1+x}.as = \frac{11}{17}.\frac{13+y}{1+y}.as,$

$$\text{or}\quad \frac{13+x}{1+x} = \frac{11}{17}.\frac{13+y}{1+y} \qquad \text{.....(i)}$$

If twice as much gold had been added to the same quantity of silver, the proportions of gold to silver in the two mixtures would have been $1 : \frac{x}{2}$ and $1 : \frac{y}{2}$; and by precisely the same process as above, we should obtain

$$\frac{13+\frac{x}{2}}{1+\frac{x}{2}} = \frac{7}{11}.\frac{13+\frac{y}{2}}{1+\frac{y}{2}} \qquad \text{..... (ii)}$$

From (i), we find $y = \frac{21x - 13}{35 + x}$,

and from (ii), $y = \frac{40x - 52}{x + 68}$.

Equating these two values of y and reducing, we get

$$19x^2 - 67x = 936,$$

whence $x = 9$, and $\therefore\ y = 4$.

Hence 1 : 9, and 1 : 4 are the required proportions.

RATIO, PROPORTION, AND VARIATION.

1. $\frac{a + 2b}{a + b}$ is the greater. (Art. 84).

2. $$\frac{\sqrt{3}}{\sqrt{2}} \text{ is } > \text{ or } < \frac{49}{40},$$

according as $\frac{3}{2}$ is $>$ or $< \left(\frac{49}{40}\right)^2$,

..... 3×40^2 is $>$ or $< 2 \times 49^2$,

..... 4800 is $>$ or $<$ 4802.

But 4802 is $>$ 4800, therefore $\frac{49}{40}$ is the greater of the given ratios.

3. Let x be the required quantity, then

$$a + x : b + x :: c : d,$$

whence $x = \frac{ad - bc}{c - d}$.

4. Let a, b, c, d be the four quantities of which a is the greatest, and consequently d the least, then

$$\frac{a}{b} = \frac{c}{d}, \text{ and } \therefore\ \frac{a - b}{b} = \frac{c - d}{d},$$

but b is $> d$, $\therefore\ a - b$ is $> c - d$, or $a + d$ is $> b + c$, i. e. the sum of the greatest and least is $>$ the sum of the other two.

5. Since $a : b :: c : d$,

$$a^m : b^m :: c^m : d^m,$$

and a^m being the greatest of these four proportionals, it follows by the last Example that $a^m + d^m$ is $> b^m + c^m$.

6. This is a particular case of Art. 90.

7. Since $y \propto x$, we have $y = Cx$, C being some constant. (Art. 95). But when $x = 1$, y must $= 3$,

$\therefore\ 3 = C$, and $\therefore\ y = 3x$: When $x = 4$ the value of y is therefore 12.

8. Since $z \propto x + y$ and $y \propto x^2$, we have

$$z = C(x + y) \text{ and } y = C'x^2, \quad \ldots\ldots\ldots\ldots\ldots(\alpha)$$

C and C' being constants (Art. 95), but when $x = 1$ we must have $y = 2$ and $z = 3$; equations (α) therefore give us

$$3 = C(1 + 2), \text{ and } 2 = C',$$

$$\text{i. e. } C = 1, \text{ and } C' = 2,$$

$$\therefore\ z = x + 2x^2.$$

9. The proof is precisely the same as that given in Art. 100; $\frac{1}{x}$ being written everywhere instead of x.

10. Since $z \propto x + y$, $\quad\therefore\ z = C_1(x + y), \quad \ldots\ldots\ldots\ldots (1)$

and $\ldots\ldots\ u \propto x - y, \quad\therefore\ u = C_2(x - y), \quad \ldots\ldots\ldots\ldots (2)$

$\ldots\ldots\ x \propto u + z, \quad\therefore\ x = C_3(u + z), \quad \ldots\ldots\ldots\ldots (3)$

$C_1 C_2 C_3$ being constants, adding (1) and (2) together

$$z + u = (C_1 + C_2)x + (C_1 - C_2)y, \quad \ldots\ldots\ldots\ldots(4)$$

and $\therefore$ by (3), $\quad x = C_3(C_1 + C_2)x + C_3(C_1 - C_2)y,$

$$\text{or } \quad y = \frac{1 - C_3(C_1 + C_2)}{C_3(C_1 - C_2)}\, x,$$

$$\text{i. e. } \quad y \propto x.$$

There is an exception to this conclusion, when (1), (2), and (3) are not independent equations; which will be the case when (4) is the same equation as (3): or when $C_1 = C_2 = \frac{1}{2C_3}$.

11. See Art. 99.

12. Since $z \propto y$ and $y \propto x$, y and z each vary as x: we have therefore $y = mx$ and $z = nx$, m and n being constants:

$$\text{hence } x + y + z = (1 + m + n)\,x,$$

$$\text{and } \sqrt{yz} + \sqrt{xz} + \sqrt{xy} = \sqrt{mnx^2} + \sqrt{nx^2} + \sqrt{mx^2}$$

$$= (\sqrt{mn} + \sqrt{n} + \sqrt{m})\,x,$$

$$\therefore\ x + y + z = \frac{1 + m + n}{\sqrt{mn} + \sqrt{n} + \sqrt{m}}\,(\sqrt{yz} + \sqrt{xz} + \sqrt{xy}),$$

$$\text{i. e. } x + y + z \propto \sqrt{yz} + \sqrt{xz} + \sqrt{xy}.$$

13. Since the weight of a solid sphere $\propto$ (radius)^3,

let the weight of a sphere of radius $r = Cr^3$.

Then calling the inner and outer radii of the shell r and r', we have,

the weight of a solid sphere of radius $r' = Cr'^3$,

and ..... $r = Cr^3$,

$\therefore$ the weight of the hollow sphere $= C(r'^3 - r^3)$,

$$\therefore\ C(r'^3 - r^3) = \tfrac{7}{8}\,Cr'^3,$$

$$\text{or } r'^3 - r^3 = \tfrac{7}{8}\,r'^3,$$

$$\text{whence } r : r' :: 1 : 2.$$

14. It is manifest that if any quantity be taken from A, $\frac{2}{5}$ths of it will be water, and $\frac{3}{5}$ths wine. Also of any quantity taken from B, $\frac{3}{10}$ths will be water, and $\frac{7}{10}$ths wine.

Let the proposed mixture be made by x gallons from A, and y gallons from B, then

$\frac{2}{5}x + \frac{3}{10}y =$ the number of gallons of water $= 5$..... (i)

and $\frac{3}{5}x + \frac{7}{10}y =$ ..... wine $= 11$.....(ii)

(i) and (ii) give $x = 2$ and $y = 14$.

ARITHMETICAL PROGRESSION.

1. Let a be the first term of any arithmetical progression, d the common difference.

Then, the sum of the latter half of $2n$ terms

= the $(n+1)^{\text{th}}$ term + the $(n+2)^{\text{th}}$ term + &c. ... to n terms,

$= (a + nd) + \{a + (n+1)\,d\}$ + &c. ... to n terms,

$= \{2(a + nd) + (n-1)\,d\}\,\frac{n}{2}$,

$= \{2a + (3n-1)\,d\}\,\frac{n}{2}$,

$= \frac{1}{3}\{2a + (3n-1)\,d\}\,\frac{3n}{2}$,

$= \frac{1}{3}$ the sum of $3n$ terms of the same series.

2. $2 + 5 + 8 + \ldots$ to 11 terms $= 187$.

$3 + 7 + 11 + \ldots$ to 12 terms $= 300$.

$m + (m+1) + (m+2) + \ldots$ to m terms $= \{2m + (m-1)\}\,\frac{m}{2}$

$$= \frac{m(3m-1)}{2}.$$

$3 + 2 + 1 + \ldots$ to 8 terms $= -4$.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \ldots$ to 10 terms $= -\frac{5}{2}$.

$\frac{1}{3} + \frac{5}{6} + \frac{4}{3} + \ldots$ to n terms $= \frac{n}{12}(3n+1)$.

$2\frac{1}{2} + 2 + \frac{3}{2} + \ldots$ to n terms $= \frac{n}{4}(11-n)$.

$(a+x)^2 + (a^2+x^2) + (a-x)^2 + \ldots$ to n terms $= n(a+x)^2 - n^2 ax$.

$3 + 1 - 1 - \ldots\ldots$ to 10 terms $= -60$.

$54 + 51 + 48 + \ldots$ to 9 terms $= 378$.

$6 + \frac{11}{2} + 5 + \ldots\ldots$ to 25 terms $= 0$.

$$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \ldots \text{ to } n \text{ terms} = \frac{n-1}{2}.$$

3. Putting $S = 1455$, $a = 5$, and $n = 30$ in the formula in Art. 102, we have

$$1455 = (10 + 29d)\,15,$$

whence $d = 3$.

4. Let x be the common difference of the means: then, 14 must be the fifth term of an arithmetical series whose first term is 2, and common difference x; hence

$$14 = 2 + 4x,$$

$$\text{whence } x = 3,$$

therefore the required means are 5, 8 and 11.

5. Let M and N be the m^{th} and n^{th} terms of the series, x and y the first term and the common difference,

$$\text{then } x + (m-1)\,y = \text{the } m^{\text{th}} \text{ term} = M, \quad \ldots\ldots\ldots \text{(i)}$$

$$x + (n-1)\,y = \text{the } n^{\text{th}} \text{ term} = N. \quad \ldots\ldots\ldots \text{(ii)}$$

From (i) and (ii), we find

$$x = \frac{(m-1)\,N - (n-1)\,M}{m-n}, \text{ and } y = \frac{M-N}{m-n}.$$

Hence the sum of p terms

$$= \left\{2x + (p-1)\,y\right\}\frac{p}{2}$$

$$= \frac{p}{2} \cdot \frac{2(m-1)\,N - 2(n-1)\,M + (p-1)\,(M-N)}{m-n}$$

$$= \frac{p}{2} \cdot \frac{(2m-p-1)\,N - (2n-p-1)\,M}{m-n}.$$

6. Let S_r denote the sum of the r^{th} series, the first term of which is r, and the common difference $2r-1$,

$$\text{then } S_r = \left\{2r + (n-1)\,(2r-1)\right\}\frac{n}{2}$$

$$= \left\{2nr - (n-1)\right\}\frac{n}{2}$$

$$= n^2 r - \frac{n\,.\,(n-1)}{2};$$

from this formula, we find S_1, S_2, S_3 ... by putting r successively equal to 1, 2, 3 ...

Making these substitutions and taking the sum of the results, we have

$$S_1 + S_2 + S_3 + \ldots + S_p = n^2 (1 + 2 + 3 + \ldots + p) - \frac{n.(n-1)}{2} . p$$

$$= n^2 . \frac{p(p+1)}{2} - \frac{n(n-1)}{2} . p$$

$$= (np + n - n + 1) \frac{np}{2} = (np + 1) \frac{np}{2}.$$

7. Put $-n = n'$, then

$$S = \{2a + (n-1)d\} \frac{n}{2}$$

$$= \{2a - (n'+1)d\} \frac{-n'}{2}$$

$$= \{-2a + (n'+1)d\} \frac{n'}{2}$$

$$= \{2(d-a) + (n'-1)d\} \frac{n'}{2},$$

which is the sum of n' (i. e. $-n$) terms of a series having the common difference d, and $d - a$ for its first term.

8. Let x be the common difference of the means,

then $x = \frac{30}{n+1}$. (Art. 102. Ex. 3).

Hence the 7^{th} mean $= 1 + 7x = 1 + \frac{210}{n+1}$,

and the $(n-1)^{\text{th}}$ ..... $= 1 + (n-1)x = 1 + \frac{30(n-1)}{n+1}$,

therefore by the question,

$$1 + \frac{210}{n+1} : 1 + \frac{30(n-1)}{n+1} :: 5 : 9,$$

whence $n = 14$.

9. Let $x - y$, x and $x + y$ be the three numbers,

then $3x$ = their sum = 15, ... (i)

and $x(x^2 - y^2)$ = their product = 120..... (ii)

(i) gives $x = 5$, and (ii) then gives $y = 1$,

therefore 4, 5 and 6 are the required numbers.

10. Ans. -7, -5, -3, -1, 1, 3, 5, 7, 9, &c.

11. Let x and y be the first term and the common difference of an arithmetical progression of which a, b and c are the p^{th}, q^{th} and r^{th} terms; then we must have

$$a = x + (p - 1)y,$$
$$b = x + (q - 1)y,$$
$$c = x + (r - 1)y.$$

As we have three equations and only two unknown quantities, some equation of condition must hold between a, b, c, p, q and r in order that the equations may all be true. Multiplying the first of the above equations by $(q - r)$, the second by $(r - p)$, and the third by $(p - q)$ and adding the results, we find

$$a(q - r) + b(r - p) + c(p - q) = 0,$$

the required relation.

12. Let x, $x + 1$, $x + 2$ and $x + 3$ be the numbers,

then $x(x + 1)(x + 2)(x + 3) = 120$,

$\therefore\ x^4 + 6x^3 + 11x^2 + 6x - 120 = 0$,

or $(x - 2)(x^3 + 8x^2 + 27x + 60) = 0$,

$\therefore\ x = 2$,

and the four numbers are 2, 3, 4, 5.

13. Let x be the first term,

y the common difference,

then l being the n^{th} term, and s the sum of n terms, we have

$$l = x + (n - 1)y, \quad \text{..... (i)}$$

$$s = (x + l)\frac{n}{2} \quad \text{.....(ii)}$$

From (ii), $x = \frac{2s}{n} - l,$

and from (i), $y = \frac{l - x}{n - 1} = \frac{2(ln - s)}{n(n-1)},$

the first term and the common difference being determined, the series is known.

14. Let δ be the common difference of a series whose first term is unity, and which fulfils the proposed condition.

Then the sum of n terms

$$= \{2 + (n-1)\delta\}\frac{n}{2} = S_1,$$

and the sum of n terms beginning with the $(n+1)^{\text{th}}$

$$= \{2 + 2n\delta + (n-1)\delta\}\frac{n}{2} = S_2,$$

$$\therefore \frac{S_1}{S_2} = \frac{2 + (n-1)\delta}{2 + (3n-1)\delta} = c \text{ for all values of } n,$$

$$\therefore (2 - \delta)(c - 1) + (3c - 1)n\delta = 0 \text{ for all values of } n,$$

which can only be the case when

$$2 - \delta = 0, \text{ and } 3c - 1 = 0,$$

$$\therefore \delta = 2, \text{ or the series must be } 1, 3, 5, 7, \ldots$$

and $c = \frac{1}{3}$, and therefore the required ratio is 1 : 3.

15. The n^{th} term being $\frac{3n-1}{6}$, the *first* term is found by putting n equal to 1, the *second* term by putting n equal to 2: and so on.

Hence the series is

$$\frac{3-1}{6}, \quad \frac{3\times 2 - 1}{6}, \quad \frac{3\times 3 - 1}{6} \ldots$$

$$\text{or } \tfrac{1}{3}, \ \tfrac{5}{6}, \ \tfrac{8}{6} \ldots$$

and the sum of n terms $= \frac{n(3n+1)}{12}$.

16. If in the expression for the sum of an arithmetical series we put n equal to 1, we obtain the *first term*: hence the ratio of the first terms of the two series mentioned in the question is found by putting $n = 1$ in the given ratio. The required ratio is therefore

$$13 - 7 : 1 + 3, \text{ or } 6 : 4, \text{ or } 3 : 2.$$

17. This Example is the same with Ex. 5, p. 40.

18. Let the p^{th} term of the first series be equal to the q^{th} term of the second; then

$$2 + (p - 1)\,3 = 3 + (q - 1)\,4,$$

$$\text{whence } \frac{p}{q} = \frac{4}{3},$$

the restrictions to the values of p and q are that they shall always be in the ratio of 4 to 3, and that neither of them shall exceed 100: there are manifestly 25 such pairs of numbers, and therefore there are 25 terms identical in the two series.

19. Let x = the required number of days;

then, the distance gone over by $A = 3 + 5 + 7 + \ldots$ to x terms

$$= x(x + 2),$$

and ..... $B = 4 + 6 + 8 + \ldots$ to x terms

$$= x(x + 3),$$

the whole distance gone over $= 2x^2 + 5x$

$$= 168 \text{ by the question.}$$

Hence $x = 8$.

GEOMETRICAL PROGRESSION.

1. $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ to 12 terms

$$= \frac{\left(\frac{1}{2}\right)^{12} - 1}{\frac{1}{2} - 1} = \left(1 - \frac{1}{2^{12}}\right) \times 2 = 2 - \frac{1}{2^{11}}.$$

$$2 + \frac{1}{5} + \frac{1}{50} + \ldots \text{ to } n \text{ terms} = 2 \cdot \frac{\left(\frac{1}{10}\right)^n - 1}{\frac{1}{10} - 1} = \frac{20}{9}\left(1 - \frac{1}{10^n}\right).$$

$$\sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} + \frac{2}{3}\sqrt{\frac{2}{3}} + \ldots \text{ to } \infty = \sqrt{\frac{3}{2}} \cdot \frac{1}{1 - \frac{2}{3}} = 3\sqrt{\frac{3}{2}}.$$

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\ldots \text{ to } \infty = \frac{1}{9}.$$

$$1 - \frac{1}{5} + \frac{1}{25} - \ldots \text{ to 10 terms} = \frac{\left(-\frac{1}{5}\right)^{10} - 1}{-\frac{1}{5} - 1} = \frac{5}{6}\left(1 - \frac{1}{5^{10}}\right).$$

$$\ldots\ldots\ldots\ldots\ldots \text{ to } \infty \quad \ldots = \frac{1}{1 - \left(-\frac{1}{5}\right)} = \frac{5}{6}.$$

$$\sqrt{\frac{3}{5}} - \sqrt{6} + 2\sqrt{15} - \ldots \text{ to 8 terms}$$

$$= \sqrt{\frac{3}{5}} \cdot \frac{(-\sqrt{10})^8 - 1}{-\sqrt{10} - 1} = -\sqrt{\frac{3}{5}} \cdot \frac{10^4 - 1}{\sqrt{10} + 1}.$$

$$\frac{2}{5} - \sqrt{\frac{2}{5}} + 1 - \ldots \text{ to 8 terms}$$

$$= \frac{2}{5} \cdot \frac{\left(-\sqrt{\frac{5}{2}}\right)^8 - 1}{-\sqrt{\frac{5}{2}} - 1} = -\frac{2}{5} \cdot \frac{\left(\frac{5}{2}\right)^4 - 1}{\sqrt{\frac{5}{2}} + 1}.$$

2. Let $\frac{x}{y}$, x and xy be the three numbers, then

$$x\left(\frac{1}{y} + 1 + y\right) = 14, \quad \ldots\ldots\ldots\ldots\ldots \text{ (i)}$$

$$\text{and } x^2\left(\frac{1}{y^2} + 1 + y^2\right) = 84. \quad \ldots\ldots\ldots\ldots\ldots \text{ (ii)}$$

Squaring each side of (i), we get

$$x^2\left(\frac{1}{y^2}+1+y^2\right)+2x^2\left(\frac{1}{y}+1+y\right)=196,$$

or by (i) and (ii),

$$84+28x=196,$$

whence $x=4$, and therefore by (i) or (ii), $y=2$,

therefore the required numbers are 2, 4 and 8.

3. $.111\ldots=\frac{1}{9}$ (Art. 106) $=(\frac{1}{3})^2=(.333\ldots)^2$.

4. Let r = the common ratio of the terms,

then, the n^{th} term $=r^{n-1}$, and the sum of all that follow

$$=r^n+r^{n+1}+\ldots \text{ to } \infty = r^n.\frac{1}{1-r},$$

$$\therefore\ r^{n-1}=r^n.\frac{1}{1-r},\ \text{or } 1=\frac{r}{1-r},\ \text{or } r=\tfrac{1}{2},$$

therefore the series is $1, \frac{1}{2}, \frac{1}{4}, \ldots$

5. Let x and y be the arithmetical and geometrical means between a and b, then $x-a=b-x$, and $\frac{y}{a}=\frac{b}{y}$; whence $x=\frac{a+b}{2}$, and $y=\sqrt{ab}$. Now $(\sqrt{a}-\sqrt{b})^2$ is essentially positive, and $\therefore >0$,

$$\therefore\ a-2\sqrt{ab}+b \text{ is } >0,\ \text{or } \frac{a+b}{2}>\sqrt{ab};$$

i. e. the arithmetical mean between a and b is greater than the geometrical.

6. Let $a, ar, ar^2 \ldots$ be the quantities in geometrical progression; then, their differences are

$$ar-a,\quad ar^2-ar,\quad ar^3-ar^2,\ \ldots$$

$$\text{or } (ar-a),\quad (ar-a)r,\quad (ar-a)r^2,\ \ldots$$

which are manifestly in geometrical progression.

7. Let $\frac{x}{y}$, x and xy be the three numbers, then

$$x\left(\frac{1}{y}+1+y\right)=13, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{and } \frac{x\left(\frac{1}{y}+1\right)}{x(1+y)}=\frac{1}{3}, \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

(ii) gives $y = 3$, and (i) then gives $x = 3$; the required numbers are therefore 1, 3, 9.

8. Let x and $x + 48$ be the required numbers; then

the arithmetical mean between them $= x + 24$,

... geometrical ..... $= \sqrt{x^2 + 48x}$,

therefore by the question, $x + 24 = \sqrt{x^2 + 48x} + 18$,

whence $x = 1$; and the required numbers are therefore 1 and 49.

9. Let a be the first term of the series, r the common ratio of the terms; then

the sum of the odd terms $= a + ar^2 + ar^4 + \ldots$

..... even ..... $= ar + ar^3 + ar^5 + \ldots$

$= r(a + ar^2 + ar^3 + \ldots)$

$= r$ (the sum of the odd terms),

therefore the sum of the odd terms : the sum of the even :: 1 : r.

10. Let $\frac{x}{y}$, x and xy be the required numbers, a their sum, and b the sum of their reciprocals; then

$$x\left(\frac{1}{y}+1+y\right)=a, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{and } \frac{1}{x}\left(y+1+\frac{1}{y}\right)=b \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Hence $x = \pm\sqrt{\frac{a}{b}}$, and $y = \frac{-1 \pm \sqrt{ab} + \sqrt{ab \mp 2\sqrt{ab} - 3}}{2}$,

and therefore $\frac{x}{y}$, x and xy are known.

11. Let the series be a, ar, ar^2, ... to $(2n+1)$ terms, then

the sum of the squares of the terms $= a^2 . \frac{(r^2)^{2n+1} - 1}{r^2 - 1}$

$$= a \frac{r^{2n+1} - 1}{r - 1} . \left(a \frac{r^{2n+1} + 1}{r + 1} \right)$$

$$= a \frac{r^{2n+1} - 1}{r - 1} \left\{ a (1 - r + r^2 - r^3 + \ldots - r^{2n-1} + r^{2n}) \right\}$$

$$= a \frac{r^{2n+1} - 1}{r - 1} \left\{ (a + ar^2 + ar^4 + \ldots + ar^{2n}) - (ar + ar^3 + \ldots + ar^{2n-1}) \right\}$$

= the sum of all the terms multiplied by the excess of the odd terms above the even.

12. Let x, xy, xy^2 and xy^3 be the required numbers, then, by the question,

$$xy^2 - xy = 2, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and $$xy^3 - x = 7; \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

dividing (ii) by (i), we get

$$\frac{y^3 - 1}{y^2 - y} = \frac{7}{2}, \text{ or } \frac{y^2 + y + 1}{y} = \frac{7}{2},$$

whence $y = 2$, and therefore $x = 1$.

The required numbers are therefore 1, 2, 4, 8.

13. Let the series be

$$x, xy, xy^2 \ldots$$

then $xy + xy^2 = 24$, and $xy^3 + xy^4 = 216$,

these equations give $x = 2$.

14. See Ex. 4, p. 36.

15. We have $S_p = \dfrac{1}{1 - \dfrac{1}{r^p}}$, and $\therefore \dfrac{1}{S_p} = 1 - \dfrac{1}{r^p}$.

Hence $\dfrac{1}{S_1} + \dfrac{1}{S_2} + \ldots + \dfrac{1}{S_n}$

$= \left(1 - \dfrac{1}{r}\right) + \left(1 - \dfrac{1}{r^2}\right) + \left(1 - \dfrac{1}{r^3}\right) + \ldots + \left(1 - \dfrac{1}{r^n}\right)$

$= (1 + 1 + 1 + \ldots \text{ to } n \text{ terms}) - \left(\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \ldots \text{ to } n \text{ terms}\right)$

$= n - \dfrac{r^n - 1}{r^n (r - 1)}$.

16. Let a be the first term of the series,

r the common ratio of the terms,

then $P = ar^{p+q-1}$, and $Q = ar^{p-q+1}$,

hence $PQ = a^2 r^{2p-2}$,

or $\sqrt{PQ} = ar^{p-1} =$ the p^{th} term,

also $\dfrac{P}{Q} = r^{2q}$, or $r = \left(\dfrac{P}{Q}\right)^{\frac{1}{2q}}$;

$\therefore ar^{q-1} = ar^{p-1} \cdot r^{q-p}$

$= (PQ)^{\frac{1}{2}} \cdot \left(\dfrac{P}{Q}\right)^{\frac{q-p}{2q}}$

$= P^{1-\frac{p}{2q}} \cdot Q^{\frac{p}{2q}} =$ the q^{th} term.

17. Let r be the common ratio of b to a, c to b, and d to c, then

$b = ra, \quad c = rb, \quad d = rc,$

$\therefore (a + b + c + d)^2 = (a + b)^2 + (c + d)^2 + 2(a + b)(c + d)$

$= (a + b)^2 + (c + d)^2 + 2\left(\dfrac{b}{r} + \dfrac{c}{r}\right)(rb + rc)$

$= (a + b)^2 + (c + d)^2 + 2(b + c)^2$.

18. Let a be the first term, r the common ratio of the terms, then

$$S = a\frac{r^n - 1}{r - 1}, \quad S' = \frac{1}{ar^{n-1}} \cdot \frac{r^n - 1}{r - 1},$$

$$\text{and } \therefore \frac{S}{S'} = a^2 r^{n-1}.$$

$$\text{Also } P = a \,.\, ar \,.\, ar^2 \,.\, ar^3 \ldots \text{ to } n \text{ factors}$$

$$= a^n \,.\, r^{1+2+3+\ldots \text{ to } n \text{ terms}} = a^n r^{\frac{n.(n-1)}{2}}$$

$$= (a^2 r^{n-1})^{\frac{n}{2}} = \left(\frac{S}{S'}\right)^{\frac{n}{2}}.$$

HARMONICAL PROGRESSION.

1. Let G and H be the geometrical and harmonical means between x and y; then

$$G = \sqrt{xy} \quad \text{(Ex. 5, p. 46)},$$

$$\text{and } x : y :: x - H : H - y \text{ (Art. 108), or } H = \frac{2xy}{x + y}.$$

$$\text{But } \frac{G^2}{H^2} = \frac{1}{n^2}, \quad \therefore \frac{(x+y)^2}{4xy} = \frac{1}{n^2}, \quad \ldots\ldots\ldots \text{ (i)}$$

$$\text{and } \frac{(x+y)^2 - 4xy}{4xy} = \frac{1 - n^2}{n^2}, \text{ or } \frac{(x-y)^2}{4xy} = \frac{1 - n^2}{n^2}, \quad \ldots\text{(ii)}$$

dividing (i) by (ii), we have

$$\left(\frac{x+y}{x-y}\right)^2 = 1 - n^2,$$

$$\therefore \frac{x+y}{x-y} = \sqrt{1 - n^2}, \quad \text{and } \frac{x}{y} = \frac{1 + \sqrt{1 - n^2}}{1 - \sqrt{1 - n^2}}.$$

2. The reciprocals of quantities in harmonical progression are in arithmetical progression, and *vice versâ* (Art. 108); hence the reciprocals of the terms in continuation of the arithmetical series $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ will be the terms in the continuation

of the proposed series. The arithmetical series, of which $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ are consecutive terms, is

$$\ldots\ldots \tfrac{5}{6},\ \tfrac{2}{3},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{6},\ 0,\ -\tfrac{1}{6},\ -\tfrac{1}{3},\ -\tfrac{1}{2},\ \ldots$$

therefore the harmonical series of which 2, 3, 6 are consecutive terms, is

$$\ldots\ldots \tfrac{6}{5},\ \tfrac{3}{2},\ 2,\ 3,\ 6,\ \infty,\ -6,\ -3,\ -2,\ \ldots$$

3. Let $a_1, a_2, \ldots a_r, a_{r+1}, \ldots$ be the series of terms in harmonical progression; then $\frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a_r}, \frac{1}{a_{r+1}}, \ldots$ are in arithmetical progression;

$$\therefore \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_{r+1}} - \frac{1}{a_r},$$

$$\therefore \frac{a_1 - a_2}{a_1 a_2} = \frac{a_r - a_{r+1}}{a_r a_{r+1}},$$

$$\text{or } a_1 a_2 : a_r a_{r+1} :: a_1 - a_2 : a_r - a_{r+1},$$

which is the property enunciated.

4. Let x and y be the required numbers, then, by the question,

$$\frac{x+y}{2} - \sqrt{xy} = 13, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\sqrt{xy} - \frac{2xy}{x+y} = 12 \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

From (i), we have $\frac{x+y}{2} = 13 + \sqrt{xy}$,

$\ldots$ (ii), $\ldots\ldots\ldots$ $\frac{2xy}{x+y} = \sqrt{xy} - 12.$

Multiplying the last two equations together, we have

$$\sqrt{xy} = 12 \times 13,$$

and therefore by (i), $x + y = 2 \times 13^2$;

these equations give

$$x = 234, \text{ and } y = 104.$$

5. Let x, y, z be the arithmetic, geometric, and harmonic means between a and b; then (Ex. 5, p. 46, and Ex. 1, p. 50)

$$x = \frac{a+b}{2}, \quad y = \sqrt{ab}, \quad z = \frac{2ab}{a+b},$$

$$\therefore \frac{y}{x} = \frac{2\sqrt{ab}}{a+b}, \text{ and } \frac{z}{y} = \frac{2\sqrt{ab}}{a+b},$$

$$\therefore \frac{y}{x} = \frac{z}{y};$$

i. e. x, y and z are in geometrical progression.

6. Ans. $\frac{2}{7}$, $\frac{1}{6}$ and $\frac{2}{17}$.

7. We have $x = \frac{a+b}{2}$, $z = \sqrt{ab}$, and $y = \frac{2xz}{x+z}$,

therefore substituting for x and z in the expression for y, we have

$$y = \frac{(a+b)\sqrt{ab}}{\frac{a+b}{2} + \sqrt{ab}}$$

$$= \frac{2(a+b)}{\left(\frac{a}{b}\right)^{\frac{1}{2}} + 2 + \left(\frac{b}{a}\right)^{\frac{1}{2}}} = \frac{2(a+b)}{\left\{\left(\frac{a}{b}\right)^{\frac{1}{4}} + \left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}^2}.$$

8. Let M and N be the m^{th} and n^{th} terms of a harmonical progression; then $\frac{1}{M}$ and $\frac{1}{N}$ are the m^{th} and n^{th} terms of an arithmetical progression; let a be the first term, and δ the common difference of this series, then

$$\frac{1}{M} = a + (m-1)\delta, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{1}{N} = a + (n-1)\delta \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Multiply (i) by m, and (ii) by n, and subtract, then

$$\frac{m}{M} - \frac{n}{N} = (m-n)a + (m^2 - n^2 - m + n)\delta,$$

or $\dfrac{mN - nM}{MN} = (m - n)\{a + (m + n - 1)\delta\}$

$= (m - n)\{\text{the } (m + n)^{\text{th}} \text{ term of the series}\}$,

therefore the $(m + n)^{\text{th}}$ term of the arithmetical series

$$= \frac{1}{m - n} \cdot \frac{mN - nM}{MN},$$

and the $(m + n)^{\text{th}}$ term of the harmonical series

$$= \frac{(m - n)\,MN}{mN - nM}.$$

[Obs. The direct method of solving this problem is, to find a and δ from (i) and (ii), and substitute their values in the expression for the $(m + n)^{\text{th}}$ term of the arithmetical series.]

9. Let a, b, c, d be the four numbers, then by the data of the question,

$$a - b = b - c, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{and } b : d :: b - c : c - d, \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

or by (i), $b : d :: a - b : c - d$.

Hence $bc - bd = ad - bd$,

and $\dfrac{a}{b} = \dfrac{c}{d}$, i.e. $a : b :: c : d$.

10. If P, Q, R are the p^{th}, q^{th} and r^{th} terms of an harmonical series,

$\dfrac{1}{P}$, $\dfrac{1}{Q}$, $\dfrac{1}{R}$ are the p^{th}, q^{th} and r^{th} terms of an arithmetical series,

therefore by (Ex. 11, p. 42), we must have

$$(q - r)\frac{1}{P} + (r - p)\frac{1}{Q} + (p - q)\frac{1}{R} = 0,$$

$$\text{or } (p - q)PQ + (p - r)PR + (q - r)QR = 0.$$

PERMUTATIONS AND COMBINATIONS.

1. Ans. 720, 3024 and 720. (Art. 112).

2. Ans. 35. (Art. 113. Cor.)

3. Ans. 1260. (Art. 113. Ex.)

4. The number of different guards that can be posted is manifestly equal to the number of ways in which 4 men can be taken out of 50; or to the number of combinations of 50 things taken 4 together; which, by Art. 114,

$$= \frac{50 \cdot 49 \cdot 48 \cdot 47}{1 \cdot 2 \cdot 3 \cdot 4} = 230300.$$

Again, if there were only 49 men, we could post a guard of 4 in $\frac{49 \cdot 48 \cdot 47 \cdot 46}{1 \cdot 2 \cdot 3 \cdot 4}$ or 211876 ways, any particular man will therefore be engaged on 230300 − 211876, or 18424 different nights.

5. *Any* three angular points of the polygon can be made the angular points of a triangle; the number of triangles required will therefore be the number of ways in which we can select three points out of n, i.e. the number of combinations of n things taken 3 together, which

$$= \tfrac{1}{6} n (n-1) (n-2).$$

6. There are 21 consonants and 5 vowels in the alphabet; we can therefore take 4 consonants in $\frac{21 \cdot 20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3 \cdot 4}$ different ways; and 2 vowels in $\frac{5 \cdot 4}{1 \cdot 2}$ different ways. Each group of 4 consonants may be combined with each group of 2 vowels, we shall then have $\frac{21 \cdot 20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{5 \cdot 4}{1 \cdot 2}$ or 59850 different groups, each containing 4 consonants and 2 vowels.

7. The given person must have two of the remaining five as neighbours; and two persons can be selected from 5 in

$\frac{5 \times 4}{1 \times 2}$ or 10 different ways: also, with each pair of neighbours he can take six positions: 60 is therefore the required number of arrangements.

8. [This question should have referred to *combinations* and not to *permutations*.] Substituting successively $p+2$ and p for r, in the formula of Art. 114; and dividing the first result by the second, we get

$$\frac{(n-p)(n-p-1)}{(p+1)(p+2)} = \frac{b}{a},$$

$$\text{whence } n = p + \frac{1}{2} \pm \sqrt{\frac{1}{4} + (p+1)(p+2)\frac{b}{a}}.$$

If $a = 5$, $b = 18$, $p = 3$; we have

$$n = 3 + \frac{1}{2} \pm \sqrt{\frac{1}{4} + 4 \times 18}$$

$$= 3 + \frac{1}{2} \pm \frac{17}{2} = 12,$$

the lower sign being inadmissible.

9. The numbers of permutations and combinations of n things taken r together being represented by P_r and C_r respectively, we have

$$\frac{P_r}{P_{r-1}} = n - r + 1, \text{ and } \frac{C_r}{C_{r-1}} = \frac{n-r+1}{r}.$$

$$\text{Hence } 10 = n - r + 1, \text{ and } \frac{5}{3} = \frac{n-r+1}{r},$$

from which equations, we find $n = 15$ and $r = 6$.

10. Let n be the number of cards in the pack,

$$\text{then, } \frac{n(n-1)(n-2)}{1.2.3} = \text{the number of hands of 3,}$$

$$= 425n \text{ by the question.}$$

Hence $(n-1)(n-2) = 425 . 2 . 3$, which gives $n = 52$.

11. As in Ex. 6, p. 40, it may be shewn that the number of combinations each containing 3 consonants and a vowel is $\frac{19 \times 18 \times 17}{1 \times 2 \times 3} \times 5$; and with each of these combinations of 4 letters there may be formed $4 \times 3 \times 2 \times 1$ words; the number of words required therefore

$$= \frac{19 \times 18 \times 17}{1 \times 2 \times 3} \times 5 \times 4 \times 3 \times 2 \times 1 = 116280.$$

12. This problem can be solved with facility, by supposing the die (A) which has n faces, to present the faces marked 1, 2, ... n successively, and then finding how many faces of the other die (B) will combine in making a throw not previously introduced. Thus,

Supposing (A) to present (1),

(B) may present any one of its $(n + r)$ faces, thus making $(n + r)$ throws.

Supposing (A) to present (2),

B may present any one of its faces except that marked (1), {the throw (1, 2) having been made before}, thus making $(n + r - 1)$ throws.

Supposing (A) to present (3),

B may present any one of its faces except (1) and (2), {the throws (1, 3) and (2, 3) having been made before}, thus making $(n + r - 2)$ throws,

and so on, until when A presents (n), B may present any one of the faces n, $n + 1$, $n + 2$ ... $n + r$, thus making $r + 1$ throws not before introduced; the total number of throws is therefore

$$= (n + r) + (n + r - 1) + (n + r - 2) + \text{... to } n \text{ terms}$$

$$= nr + \frac{n(n + 1)}{2}.$$

13. The number of different hands it is possible to hold at whist is equal to the number of combinations of 52 things taken 32 together, which by the formula in Art. 114, will be found to be 635013559600.

14. Let m be any integer greater than n; then since the number of combinations of m things taken n together is necessarily an integer, the expression

$$\frac{m(m-1)\ldots(m-n+1)}{1.2.3\ldots n}$$

is necessarily an integer, or $m(m-1)\ldots(m-n+1)$ is divisible by $1.2.3\ldots n$: i. e. the product of any n consecutive integers is divisible by $1.2.3\ldots n$.

BINOMIAL THEOREM.

1. $(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

$(a + bx + cx^2)^4$

$= a^4 + 4a^3bx + (4a^3c + 6a^2b^2)x^2 + (12a^2bc + 4ab^3)x^3$

$+ (6a^2c^2 + 12ab^2c + b^4)x^4 + (12abc^2 + 4b^3c)x^5$

$+ (4ac^3 + 6b^2c^2)x^6 + 4bc^3x^7 + c^4x^8.$

$(1 + x\sqrt{2})^6$

$= 1 + 6\sqrt{2}x + 30x^2 + 40\sqrt{2}x^3 + 60x^4 + 24\sqrt{2}x^5 + 8x^6.$

$(\sqrt{a} + \sqrt{x})^{10}$

$= a^5 + 10a^{\frac{9}{2}}x^{\frac{1}{2}} + 45a^4x + 120a^{\frac{7}{2}}x^{\frac{3}{2}} + 210a^3x^2 + 252a^{\frac{5}{2}}x^{\frac{5}{2}}$

$+ 210a^2x^3 + 120a^{\frac{3}{2}}x^{\frac{7}{2}} + 45ax^4 + 10a^{\frac{1}{2}}x^{\frac{9}{2}} + x^5.$

2. $(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}\frac{b}{a^{\frac{1}{2}}} - \frac{1}{8}\frac{b^2}{a^{\frac{3}{2}}} + \frac{1}{16}\frac{b^3}{a^{\frac{5}{2}}} - \frac{5}{128}\frac{b^4}{a^{\frac{7}{2}}} + \ldots$

$$(1+x)^{\sqrt{2}} = 1 + \sqrt{2}x + \frac{\sqrt{2}(\sqrt{2}-1)}{1.2}x^2 + \frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)}{1.2.3}x^3$$

$$+ \frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)(\sqrt{2}-3)}{1.2.3.4}x^4 + \ldots$$

$$(1-x)^{\frac{1}{3}} = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 - \frac{10}{243}x^4 - \dots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \dots$$

3. Let $(1+x)^n = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n$,

then, putting $x = 1$, we have

$2^n = a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n$ = the sum of the coefficients.

Also, putting $x = -1$, we have

$$0 = a_0 - a_1 + a_2 - \dots + (-1)^{n-1}a_{n-1} + (-1)^n a_n,$$

$$\text{or } a_0 + a_2 + a_4 \dots = a_1 + a_3 + a_5 + \dots$$

i.e. the sum of the coefficients of the odd terms equals the sum of the coefficients of the even, equals half the sum of all the coefficients $= 2^{n-1}$.

4. Let $(1+x)^n = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n$, ... (i)

then, since n is an integer, we have (Art. 118. Cor. 2)

$$(x+1)^n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n, \dots \text{(ii)}$$

(i) and (ii) are *identical* equations; their *second* sides merely express their *first* sides in another form: hence, the product of their second sides will be precisely the same as that of their first sides. Equating therefore the coefficient of x^n in the product of the second sides, with that in the product of the first sides, we have

$$a_0^2 + a_1^2 + a_2^2 + \dots + a_n^2$$

= the coefficient of x^n in the expansion of $(1+x)^{2n}$

$$= \frac{2n(2n-1)\dots(2n-n+1)}{1.2\dots n} = \frac{1.2.3\dots 2n}{(1.2.3\dots n)^2}.$$

5. The term required is the $\left(\frac{n}{2}+1\right)^{\text{th}}$, which

$$= \frac{n(n-1)\ldots(n-\frac{n}{2}+1)}{1.2\ldots\frac{n}{2}}$$

$$= \frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\frac{n}{2}} = \frac{1.2.3\ldots\left(\frac{n}{2}+1\right)}{\left(1.2.3\ldots\frac{n}{2}\right)^2}.$$

6. Ans. $-\frac{1}{2}\frac{x^3}{(3a)^{\frac{5}{3}}}$; $-\frac{154}{9}\frac{x^6}{(2a)^{\frac{17}{3}}}$;

$$\frac{1.6.11.16.21.26.31}{1.2.3.4.5.6.7}\left(\frac{x}{5}\right)^7.$$

7. The $(p+1)^{\text{th}}$ coefficient of $(a+b)^n$, is derived from the p^{th} by multiplying it by $\frac{n-p+1}{p}$: the p^{th} coefficient will therefore be $>$ the $(p+1)^{\text{th}}$ if this factor be <1, or if

$$n-p+1 \text{ be } < p, \text{ or if } \frac{n+1}{2} < p, \text{ or if } p > \frac{n+1}{2}.$$

8. We have

$$\frac{n(n-1)\ldots(n-p+2)}{1.2\ldots(p-1)} = P, \text{ and } \frac{n(n-1)\ldots(n-p+1)}{1.2\ldots p} = P';$$

dividing the latter of these equations by the former, we have

$$\frac{n-p+1}{p} = \frac{P'}{P}; \text{ whence, } n = p\left(1+\frac{P'}{P}\right) - 1.$$

9. [Obs. When x is said to be *very small* when compared with any other quantity, it is supposed to be so small that its square may be left out of account when compared with the square of that other quantity].

Expanding and neglecting x^2, x^3, &c., we have

$$\frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{(1+x)+(1+x)^{\frac{1}{2}}} = \frac{1+\frac{1}{2}x+1-\frac{2}{3}x}{1+x+1+\frac{1}{2}x}, \text{ nearly} = \frac{2-\frac{x}{4}}{2+\frac{3x}{2}}$$

$$= \left(1-\frac{x}{12}\right)\left(1+\frac{3x}{4}\right)^{-1} = \left(1-\frac{x}{12}\right)\left(1-\frac{3x}{4}\right), \text{ nearly}$$

$$= 1-\frac{5x}{6} \text{ nearly.}$$

10. The series after the first n terms

$$= \frac{m(m-1)\ldots(m-n+2)(m-n+1)}{1.2.3\ldots(n-1)n}(-x)^n$$

$$+ \frac{m(m-1)\ldots(m-n+2)(m-n+1)(m-n)}{1.2\ldots(n-1)n(n+1)}(-x)^{n+1}+\ldots$$

which,

since $N = \frac{m(m-1)\ldots(m-n+2)}{1.2.3\ldots(n-1)}(-x)^{n-1}$, is equal to

$$N.\frac{m-n+1}{n}(-x) + N.\frac{m-n+1}{n}.\frac{m-n}{n+1}(-x)^2 + \ldots$$

$$= Nx\left(1-\frac{m+1}{n}\right) + Nx^2\left(1-\frac{m+1}{n}\right)\left(1-\frac{m+1}{n+1}\right)+\ldots$$

11. Solving the equation $z^2+z+1=0$, we have

$$z = \frac{-1\pm\sqrt{-3}}{2}; \text{ whence } z^2 = \frac{-1\mp\sqrt{-3}}{2} \text{ and } z^3 = 1;$$

also, $z^{3p} = (z^3)^p = 1$, $z^{3p+1} = z^{3p}.z = z$,

and $z^{3p+2} = z^{3p}.z^2 = z^2$,

$$\left.\begin{aligned} (z^{3p})^2 + z^{3p} + 1 &= 3, \\ (z^{3p+1})^2 + z^{3p+1} + 1 &= z^2 + z + 1 = 0, \\ (z^{3p+2})^2 + z^{3p+2} + 1 &= z + z^2 + 1 = 0. \end{aligned}\right\} (A)$$

These equations shew that if r is a multiple of 3, the expression $x^{2r} + x^r + 1$ is equal to 3, but if r is not a multiple of 3, it is equal to zero.

The given expression becomes on expansion

$$\tfrac{1}{3}\left\{3 + nx\,\frac{1+x+x^2}{x} + \frac{n(n-1)}{1.2}x^2\,\frac{1+x^2+x^4}{x^2}\right.$$

$$+ \frac{n(n-1)(n-2)}{1.2.3}x^3\,\frac{1+x^3+x^6}{x^3} + \ldots$$

$$\left. + \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}x^r.\,\frac{1+x^r+(x^r)^2}{x^r} + \ldots\right\}$$

$$= \tfrac{1}{3}\left\{3 + \frac{n(n-1)(n-2)}{1.2.3}x^3\,\frac{3}{1} + \frac{n(n-1)\ldots(n-5)}{1.2.3.4.5.6}x^6\,\frac{3}{1} + \ldots\right\}$$

by equations (A),

$$= 1 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \frac{n(n-1)\ldots(n-5)}{1.2.3.4.5.6}x^6 + \ldots$$

= the sum of those terms in the expansion of $(1+x)^n$ in which the index of x is a multiple of 3.

12. If C_r denote the number of combinations of n things taken r together, we have

$$(1+x)^n = 1 + C_1x + C_2x^2 + \ldots + C_rx^r + \ldots + C_nx^n,$$

therefore putting $x = 1$, and subtracting 1 from each side,

$$2^n - 1 = C_1 + C_2 + \ldots + C_r + \ldots + C_n$$

= the number of all possible combinations of n things.

13. Let

$$(1-x)^{-\frac{1}{2}} = 1 + A_1x + \ldots + A_{m-1}x^{m-1} + A_mx^m + \ldots$$

then, putting $x = x(1-x)$, we have

$$(1 - x + x^2)^{-\frac{1}{2}} = 1 + A_1x(1-x) + \ldots$$

$$+ A_{m-1}x^{m-1}(1-x)^{m-1}\,A_mx^m(1-x)^m + \ldots$$

Now the coefficient of x^m in

$$A_m x^m (1-x)^m = A_m,$$

$$\ldots\ldots\ldots\ldots A_{m-1} x^{m-1}(1-x)^{m-1} = -(m-1) A_{m-1},$$

$$\ldots\ldots\ldots\ldots A_{m-2} x^{m-2}(1-x)^{m-2} = (m-2)\frac{m-3}{2} A_{m-2},$$

$$\ldots\ldots\ldots\ldots A_{m-3} x^{m-3}(1-x)^{m-3} = -(m-3)\cdot\frac{m-4}{2}\cdot\frac{m-5}{2} A_{m-3},$$

&c. &c.

therefore the required coefficient

$$= A_m - (m-1) A_{m-1} + (m-2)\frac{m-3}{2} A_{m-2}$$

$$- (m-3)\frac{m-4}{2}\cdot\frac{m-5}{3} A_{m-3} + \&c.$$

The value of A_r being

$$\frac{1.3.5\ldots(2r-1)}{1.2.3\ldots r}\cdot\frac{1}{2^r}.$$

14. Expanding $(a+b)^n$ and $(a-b)^n$, and taking the sum and difference of the results, we find that

twice the *sum* of the odd terms in the expansion of $(a+b)^n$

$$= (a+b)^n + (a-b)^n, \ldots\ldots\ldots\ldots \text{(i)}$$

and twice their *difference*

$$= (a+b)^n - (a-b)^n \ldots\ldots\ldots\ldots \text{(ii)}$$

Multiplying (i) and (ii) together, we have

four times the product of the sums of the odd and even terms

$$= (a+b)^{2n} - (a-b)^{2n}.$$

15. $\sqrt{8} = \sqrt{9-1}$

$$= 3\left(1-\frac{1}{9}\right)^{\frac{1}{2}} = 3\left\{1 - \frac{1}{2}\cdot\frac{1}{9} - \frac{1}{8}\cdot\frac{1}{9^2} - \frac{1}{16}\cdot\frac{1}{9^3} - \ldots\right\}$$

$$= 3\{1 - .055555 - .001543 - .000086 - \ldots\}$$

$$= 3\{1 - .057184\} = 2.8284.$$

$$\sqrt[3]{9} = \sqrt[3]{8+1}$$

$$= 2\left(1+\frac{1}{8}\right)^{\frac{1}{3}} = 2\left\{1+\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{9}\cdot\frac{1}{8^2}+\frac{5}{9^2}\cdot\frac{1}{8^3}-\ldots\right\}$$

$$= 2\left\{1+.041666-.001736+.000120-.000010\right\}$$

$$= 2\left\{1.041786-.001746\right\} = 2.08008.$$

16. Substituting for α, β, γ, &c. their values, we have

$$1+\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{3}+\ldots$$

$$= 1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\frac{n(n-1)(n-2)}{1.2.3.4}+\ldots$$

$$= \frac{1}{n+1}\left\{(n+1)+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\ldots\right\}$$

$$= \frac{1}{n+1}\left\{1+(n+1)+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\ldots-1\right\}$$

$$= \frac{1}{n+1}\left\{(1+1)^{n+1}-1\right\} = \frac{2^{n+1}-1}{n+1}.$$

17. Let $(1+x)^n = a_0+a_1x+a_2x^2+\ldots+a_{r-1}x^{r-1}+a_rx^r+\ldots$

then, multiplying each side by $1+x$, we have

$$(1+x)^{n+1} = a_0+(a_0+a_1)x+\ldots+(a_{r-1}+a_r)x^r+\ldots$$

identically: therefore the coefficient of x^r in the expansion of the first side is equal to $a_{r-1}+a_r$: which is the property enunciated.

18. Ans. $-\dfrac{1.3.5\ldots(2p-5)}{1.2.3\ldots(p-1)}\cdot\left(\dfrac{x}{2}\right)^{p-1}$.

19. Let a, b, c, d be the coefficients of the $(r-1)^{\text{th}}$, r^{th}, $(r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ terms of the expansion of $(x+y)^n$; then

$$\frac{b}{a} = \frac{n-r+2}{r-1}, \quad \frac{c}{b} = \frac{n-r+1}{r}, \text{ and } \frac{d}{c} = \frac{n-r}{r+1};$$

eliminating n between the first and second of these equations, we have

$$(r-1)\frac{b}{a} - r\frac{c}{b} = 1, \text{ whence } r = \frac{(a+b)\,b}{b^2 - ac},$$

similarly, from the second and third of the above equations, we find

$$r = \frac{(c+d)\,b}{c^2 - bd};$$

equating these two values of r, we have

$$\frac{a+b}{b^2-ac} = \frac{c+d}{c^2-bd},$$

whence $b^3c + abd - bc^2 - adc = 2(ac^2 - bd^2)$,

or $(bc + ad)(b - c) = 2(ac^2 - bd^2)$.

20. Let $u = 1 + x + 2x^2 + 3x^3 + \ldots$

$$\therefore ux = x + x^2 + 2x^3 + \ldots$$

and $u(1-x) = 1 + x^2(1 + x + x^2 + \ldots) = 1 + \frac{x^2}{1-x}$,

$$\therefore u = \frac{1}{1-x} + \frac{x^2}{(1-x)^2} = (1-x)^{-1} + x^2(1-x)^{-2},$$

and $u^2 = (1-x)^{-2} + 2x^2(1-x)^{-3} + x^4(1-x)^{-4}$.

Hence, the coefficient of x^n in u^2

$$= \begin{cases} \text{the coefficient of } x^n \text{ in } (1-x)^{-2} \\ + 2 \text{ the coefficient of } x^{n-2} \text{ in } (1-x)^{-3} \\ + \text{the coefficient of } x^{n-4} \text{ in } (1-x)^{-4} \end{cases}$$

$$= \frac{2 \cdot 3 \ldots (2+n-1)}{1 \cdot 2 \cdot 3 \ldots n} + 2 \cdot \frac{3 \cdot 4 \ldots n}{1 \cdot 2 \cdot 3 \ldots (n-2)} + \frac{4 \cdot 5 \ldots (n-1)}{1 \cdot 2 \ldots (n-4)}$$

$$= (n+1) + (n-1)\,n + \frac{(n-3)(n-2)(n-1)}{1 \cdot 2 \cdot 3} = \frac{n(n^2+11)}{6}.$$

21. From Ex. 3, p. 58, we have

$$a_0 + a_1 + a_2 + \ldots = 2^n,$$

therefore squaring

$$a_0^2 + a_1^2 + a_2^2 + \ldots + 2(a_0a_1 + a_0a_2 + \ldots) = 2^{2n},$$

and therefore, by Ex. 4, p. 58,

$$a_0a_1 + a_0a_2 + \ldots = 2^{2n-1} - \frac{1}{2} \cdot \frac{1.2.3 \ldots 2n}{(1.2.3 \ldots n)^2}.$$

22. $$\left(\frac{1+2x}{1+x}\right)^n = \left(\frac{1+x}{1+2x}\right)^{-n} = \left(\frac{1+2x-x}{1+2x}\right)^{-n}$$

$$= \left(1 - \frac{x}{1+2x}\right)^{-n},$$

and $$\left(\frac{1+x}{1-x}\right)^n = \left(\frac{1-x}{1+x}\right)^{-n} = \left(1 - \frac{2x}{1+x}\right)^{-n},$$

the expansions may now be effected by the Binomial Theorem.

23. Dividing the numerator and denominator of the given fraction by a it becomes

$$\frac{\left\{2\frac{x}{a} - \left(\frac{x}{a}\right)^2 + \frac{b^2}{a^2}\left(\frac{x}{a}\right)^2\right\}^{\frac{1}{2}}}{\left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}}}$$

$$= \frac{\left\{2(1+h) - (1+h)^2 + \frac{b^2}{a^2}(1+h)^2\right\}^{\frac{1}{2}}}{\left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}}}$$

$$= \left(\frac{1 + \frac{b^2}{a^2} + 2\frac{b^2}{a^2}h}{1 + \frac{b^2}{a^2}}\right)^{\frac{1}{2}} \text{ omitting } h^2$$

$$= (1 + \frac{2b^2}{a^2+b^2}h)^{\frac{1}{2}}$$

$$= 1 + \frac{b^2}{a^2+b^2}h, \text{ omitting } h^2.$$

24. $$\frac{a^p}{x^m} + \frac{b^q - a^p}{x^n} = \frac{b^q}{x^n}\left(1 - \frac{a^p}{b^q} + \frac{a^p}{b^q} \cdot x^{n-m}\right)$$

$$= \frac{b^q}{x^n}\left\{1 - \frac{a^p}{b^q}(1 - x^{n-m})\right\}.$$

But x being very nearly equal to 1, the quantity $1 - x^{n-m}$ is so small that its square may be neglected, and the second side of the above equation is very nearly equal to

$$\frac{b^q}{x^n}\left\{1 - (1 - x^{n-m})\right\}^{\frac{a^p}{b^q}}$$

$$= \frac{b^q}{x^n}(x^{n-m})^{\frac{a^p}{b^q}}$$

$$= b^q \cdot x^{(n-m)\frac{a^p}{b^q} - n}.$$

25. See Ex. 9, p. 59.

26. Writing $a - c$ for b, we have

$$\frac{a^2b^2}{(a^2 - a^2x^2 + b^2x^2)^{\frac{1}{2}}} = \frac{a^2(a-c)^2}{\left\{a^2 - a^2x^2 + (a-c)^2x^2\right\}^{\frac{1}{2}}}$$

$$= a^2(a^2 - 2ac)(a^2 - 2acx^2)^{-\frac{1}{2}}, \text{ (neglecting } c^2\text{)}$$

$$= (a - 2c)\left(1 - 2\frac{c}{a}x^2\right)^{-\frac{1}{2}}$$

$$= (a - 2c)\left(1 + \frac{3cx^2}{a}\right) \text{ nearly}$$

$$= a - 2c + 3cx^2 \text{ nearly.}$$

27. The coefficient of the $(p + 3)^{\text{th}}$ term is derived from that of the $(p + 1)^{\text{th}}$ term by multiplying by

$$\frac{(2n - p)(2n - p - 1)}{(p + 1)(p + 2)},$$

this fraction must therefore be equal to unity. Hence

$$(p + 1)(p + 2) = (2n - p)(2n - p - 1),$$

solving this equation with respect to p, we get

$$p = n - 1.$$

28. The middle term is the $(n+1)^{\text{th}}$, which

$$= \frac{2n\,(2n-1)\,\ldots\,(2n-n+1)}{1\,.\,2\,.\,3\,\ldots\,n}\,x^n$$

$$= \frac{2n\,(2n-1)\,(2n-2)\,\ldots\,(n+1)\,.\,n\,.\,(n-1)\,\ldots\,3\,.\,2\,.\,1}{(1\,.\,2\,.\,3\,\ldots\,n)^2}\,x^n$$

$$= \frac{2\,.\,n\,.\,(2n-1)\,.\,2\,.\,(2n-1)\,(2n-3)\ldots 5\,.\,(2\,.\,2)\,.\,3\,.\,(2\,.\,1)\,.\,1}{(1\,.\,2\,.\,3\,\ldots\,n)^2}\,x^n$$

[each even factor in the numerator being split up into two others, of which 2 is one]

$$= \frac{(2n-1)\,(2n-3)\,\ldots\,5\,.\,3\,.\,1\,.\,n\,(n-1)\,\ldots\,3\,.\,2\,.\,1}{(1\,.\,2\,.\,3\,\ldots\,n)^2}\,2^n x^n$$

$$= \frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{1\,.\,2\,.\,3\,\ldots\,n}\,2^n x^n.$$

LOGARITHMS.

1. Taking the logarithms of each side of the first group of equations, we obtain

$$x \log a + y \log b = \log c,$$

$$x \log a_1 + y \log b_1 = \log c_1,$$

$$\text{which give } x = \frac{\log b_1 \log c - \log b \log c_1}{\log a \log b_1 - \log a_1 \log b},$$

and a similar expression for y.

From the second group of equations, we obtain

$$x \log a + y \log b + z \log c = \log d,$$

and two similar simple equations between x, y and z, these give

$$x = \frac{ld(lb_1.lc_2 - lb_2.lc_1) + ld_1(lb_2.lc - lb.lc_2) + ld_2(lb.lc_1 - lb_1.lc)}{la(lb_1.lc_2 - lb_2.lc_1) + la_1(lb_2.lc - lb.lc_2) + la_2(lb.lc_1 - lb_1.lc)},$$

where ld represents $\log d$; similar expressions may be found for y and z.

2. Let P be the population at first, P_r that at the end of the r^{th} year, then, $\frac{P_r}{m}$ and $\frac{P_r}{n}$ are the numbers of births and deaths respectively in the $(r+1)^{\text{th}}$ year, and the population at the end of the $(r+1)^{\text{th}}$ year $= P_r\left(1+\frac{1}{m}-\frac{1}{n}\right)$. Hence it appears that the population is increased each year in the ratio of $1+\frac{1}{m}-\frac{1}{n}$ to 1, and therefore $P_r = P\left(1+\frac{1}{m}-\frac{1}{n}\right)^r$. If the population be *doubled* in r years $P_r = 2P$, and therefore

$$2 = \left(1+\frac{1}{m}-\frac{1}{n}\right)^r,$$

$$\text{or } \log 2 = r \log \frac{mn+n-m}{mn},$$

$$\text{which gives } r = \frac{\log 2}{\log(mn+n-m) - \log mn}.$$

3. Let $\frac{P}{n}$ be the increase of the population P in one year; then, as in the last Example, it may be shewn that if P be the original population, the population at the end of x years is $P\left(1+\frac{1}{n}\right)^x$. If then a country trebles its population in a century, we have

$$P\left(1+\frac{1}{n}\right)^{100} = 3P,$$

$$\text{or } 100 \log\left(1+\frac{1}{n}\right) = \log 3,$$

$$\therefore \log\left(1+\frac{1}{n}\right) = \frac{\log 3}{100} = .0047712 = \log(1.01105),$$

$$\therefore 1+\frac{1}{n} = 1.01105 \text{ or } \frac{1}{n} = .01105,$$

therefore the increase per million in a year $= \frac{1000000}{n} = 11050.$

4. We have

$$2 = \frac{2^{28}}{2^{27}} \cdot \frac{5^{10}}{5^3 . 5^7} \cdot \frac{3^{12}}{3^7 . 3^5} = \frac{2^{4\times 7}}{3^7 . 5^7} \cdot \frac{5^{2\times 5}}{3^5 . 2^{3\times 5}} \cdot \frac{3^{4\times 3}}{5^3 . 2^{4\times 3}}$$

$$= \left(\frac{2^4}{3\times 5}\right)^7 \cdot \left(\frac{5^2}{3\times 2^3}\right)^5 \cdot \left(\frac{3^4}{5\times 2^4}\right)^3 = \left(\frac{16}{15}\right)^7 \cdot \left(\frac{25}{24}\right)^5 \cdot \left(\frac{81}{80}\right)^3,$$

therefore taking logarithms

$$\log 2 = 7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}.$$

5. $$x \log 20 = \log 100 = 2,$$

$$\therefore\ x = \frac{2}{\log 20} = \frac{2}{1.30103} = 1.537.$$

7. Solving the equation with respect to a^x, we find

$a^x = \frac{1}{2}(1 + \sqrt{5})$; whence $x \log a = \log \frac{1}{2}(1 + \sqrt{5})$.

8. Let x be the number of factors, then

$$a^1 . a^3 . a^5 \ldots a^{2x-1} = p, \quad \text{or} \quad a^{1+3+5+\ldots \text{ to } x \text{ terms}} = p,$$

$$\text{or} \quad a^{x^2} = p; \quad \therefore\ x^2 \log a = \log p, \quad \text{or} \quad x = \sqrt{\frac{\log p}{\log a}}.$$

MISCELLANEOUS PROBLEMS.

1. Let $y^m = x$, then $y^{mn} - 1 = x^n - 1$.

$$\text{Since} \quad \frac{x^n - 1}{x - 1} = x^{n-1} + \frac{x^{n-1} - 1}{x - 1};$$

it follows that, if $x^{n-1} - 1$ be divisible by $x - 1$, $x^n - 1$ is so also: or, if $x^n - 1$ be divisible by $x - 1$ for any one value of the index, it is *proved* to be so for that index increased by unity: but $x - 1$ and $x^2 - 1$ are manifestly divisible by $x - 1$: therefore, by induction, $x^n - 1$ is divisible by $x - 1$, whatever positive integer n may be, i. e. $y^{mn} - 1$ is divisible by $y^m - 1$; and similarly, $y^{mn} - 1$ is divisible by $y^n - 1$, whatever positive integers m and n are.

2. $a^6 + a^4b^2 + a^2b^4 + b^6$ is $>$ or $< (a^3 + b^3)^2$,

according as $a^4b^2 - 2a^3b^3 + a^2b^4$ is $>$ or < 0,

..... $(a^2b - ab^2)^2$ is $>$ or < 0.

But a square is necessarily > 0: therefore the former of the given quantities is the greater.

3. $(a_1 + a_2 + \ldots + a_n)^2$

$= a_1^2 + a_2^2 + \ldots + a_n^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 + \ldots$

$= n(a_1^2 + a_2^2 + \ldots + a_n^2) - (a_1^2 - 2a_1a_2 + a_2^2)$

$- (a_1^2 - 2a_1a_3 + a_3^2) - (a_2^2 - 2a_2a_3 + a_3^2) - \&c.$

[each of the quantities a_1^2, a_2^2, ... being added and subtracted $(n-1)$ times]

$= n(a_1^2 + a_2^2 + \ldots + a_n^2) - (a_1 - a_2)^2 - (a_1 - a_3)^2 - (a_2 - a_3)^2 - \&c.$

4. By actual division, we have

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}.$$

Now, a being $> b$, a^{n-1}, $a^{n-2}b$, ... ab^{n-2} are each $> b^{n-1}$; and for the same reason $a^{n-2}b$, $a^{n-3}b$, ... b^{n-1} are each $< a^{n-1}$;

$$\therefore \frac{a^n - b^n}{a - b} \text{ is } > nb^{n-1} \text{ and } < na^{n-1}.$$

or $a^n - b^n$ is $> nb^{n-1}(a - b)$ and $< na^{n-1}(a - b)$.

5. $$n = a^2 \pm b = a^2\left(1 \pm \frac{b}{a^2}\right),$$

$$\therefore \sqrt{n} = a\left(1 \pm \frac{b}{a^2}\right)^{\frac{1}{2}}$$

$$= a\left(1 \pm \frac{1}{2}\frac{b}{a^2} - \frac{1}{8}\frac{b^2}{a^4} \pm \ldots\right)$$

$$= a \pm \frac{b}{2a}\left(1 \mp \frac{b}{4a^2}\right) \text{ nearly, } (b \text{ being small})$$

$$= a \pm \frac{b}{2a}\left(1 \pm \frac{b}{4a^2}\right)^{-1} \text{ very nearly}$$

$$= a \pm \frac{b}{2a} \cdot \frac{4a^2}{1 \pm \frac{b}{4a^2}}$$

$$= a \pm \frac{2ab}{4a^2 \pm b} \text{ very nearly.}$$

6. If n is not less than 3, it is manifest that $n(n-1)$ exceeds 2 by a quantity greater than unity, $\therefore n^2 > n+2$ by a quantity > 1, and $n^3 > n^2 + 2n$ by a quantity $> n$; therefore $n^3 - 1 > n^2 + 2n$ by a quantity $> n-1$, i.e. n^3 is $> n^2 + 2n + 1$, whence $\sqrt{n}$ is $> \sqrt[3]{n+1}$.

7. If we multiply the three expressions

$$\left.\begin{aligned} &1 + a + a^2 + \ldots + a^p, \\ &1 + b + b^2 + \ldots + b^q, \\ &1 + c + c^2 + \ldots + c^r, \end{aligned}\right\} \quad (A)$$

together, we obtain an expression in which the first term is unity, and each of the other terms of the form $a^{p'} b^{q'} c^{r'}$: now this term signifies that p' a's, q' b's, and r' c's are multiplied together; by taking all possible values of p', q', r' we shall obtain a representative of every possible combination of a's, b's and c's, the number of a's, b's and c's not exceeding p, q and r respectively. The product of the expressions (A) will give all these combinations + 1; the *number* of them will be found by putting $a = b = c = 1$.

Hence the required number of combinations + 1

$$= \begin{cases} (1 + 1 + \ldots + 1^p) \cdot (1 + 1 + \ldots + 1^q) \\ \times (1 + 1 + \ldots + 1^r) \end{cases}$$

$$= (p+1)(q+1)(r+1),$$

therefore the required number of combinations

$$= (p+1)(q+1)(r+1) - 1.$$

8. See Ex. 12, p. 61.

9. Writing $-x$ for x in Ex. 12, p. 61, we have

$$(1-x)^n = 1 - C_1x + C_2x^2 - C_3x^3 + C_4x^4 - \ldots + C_{2r}x^{2r} - C_{2r+1}x^{2r+1} + \ldots$$

therefore, putting $x = 1$, we have

$$0 = 1 - C_1 + C_2 - C_3 + C_4 - \ldots + C_{2r} - C_{2r+1} + \ldots$$

or $C_1 + C_3 + \ldots + C_{2r+1} + \ldots = 1 + C_2 + C_4 + \ldots + C_{2r} + \ldots$

Q. E. D.

10. $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} = \frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{(c+d\sqrt{-1})(c-d\sqrt{-1})} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\sqrt{-1}.$$

11. Let $$\frac{ad-bc}{a-b-c+d} = \frac{ac-bd}{a-b-d+c} = \lambda,$$

$$\therefore ad - bc = \lambda(a-b-c+d), \ldots\ldots\ldots\ldots \text{(i)}$$

and $$ac - bd = \lambda(a-b-d+c) \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Adding (i) to (ii) and dividing the result by $a-b$, we have

$$d + c = 2\lambda.$$

Similarly, by subtraction, we obtain

$$a + b = 2\lambda,$$

$$\therefore \lambda = \frac{a+b+d+c}{4}.$$

12. $$\frac{\sqrt[3]{5.12} + \sqrt[3]{.03375}}{\sqrt[3]{80} - \sqrt[3]{.01}} = \frac{\sqrt[3]{512} + \sqrt[3]{3.375}}{\sqrt[3]{8000} - \sqrt[3]{1}} = \frac{8 + 1.5}{20 - 1} = .5.$$

13. $$4Nn = (N+n)^2 - (N-n)^2,$$

$$\therefore 2\sqrt{Nn} = (N+n)\left\{1 - \frac{1}{2}\left(\frac{N-n}{N+n}\right)^2\right\} \text{ nearly,}$$

$$4\sqrt{Nn} = \frac{2(N+n)^2 - (N-n)^2}{N+n}$$

$$= \frac{4Nn + (N+n)^2}{N+n}$$

$$= \frac{4Nn}{N+n} + N + n,$$

$$\therefore \sqrt{\frac{N}{n}} = \frac{N}{N+n} + \frac{N+n}{4n}.$$

14. $x^2 - 2x + 3$ is equal to $(x-1)^2 + 2$; now, $(x-1)^2$, being a square number, cannot be negative, it will therefore give the least value of $x^2 - 2x + 3$ when it vanishes or when $x = 1$.

15. Let $S = \frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} + \ldots + \frac{a+(n-1)b}{r^n}$,

$$\therefore \frac{S}{r} = \frac{a}{r^2} + \frac{a+b}{r^3} + \ldots + \frac{a+(n-2)b}{r^n} + \frac{a+(n-1)b}{r^{n+1}}.$$

Subtracting, we have

$$S\left(1 - \frac{1}{r}\right) = \frac{a}{r} - \frac{a}{r^{n+1}} + b\left(\frac{1}{r^2} + \frac{1}{r^3} + \ldots + \frac{1}{r^{n+1}}\right) - \frac{nb}{r^{n+1}},$$

whence, $S = \left\{\frac{a}{r-1} + \frac{b}{(r-1)^2}\right\}\left(1 - \frac{1}{r^n}\right) - \frac{nb}{(r-1)r^n}$.

In the Example,

$$a = 1, \quad b = 2, \quad r = 2; \quad \text{and} \quad S = 3 - \frac{2n+3}{2^n}.$$

16. The square of a possible quantity is never negative, or never < 0, therefore $(x-y)^2$ is never < 0, or $x^2 - 2xy + y^2$ is never < 0, or $x^2 + y^2$ is never $< 2xy$.

17. $x - y$ is $>$ or $< (\sqrt{x} - \sqrt{y})^2$, according as $x - y$ is $>$ or $< x - 2\sqrt{xy} + y$, or according as $2\sqrt{xy}$ is $>$ or $< 2y$, i. e. according as x is $>$ or $< y$.

18. $(a-b)^2$ is >0, or $a^2-2ab+b^2$ is >0,

$\therefore\ a^2-ab+b^2$ is $>ab$,

and $(a^2-ab+b^2)\,(a+b)>ab\,(a+b)$,

or $a^3+b^3>ab\,(a+b)$,

and $\therefore\ \dfrac{a}{b^2}+\dfrac{b}{a^2}>\dfrac{1}{a}+\dfrac{1}{b}$.

19. Since the square of a possible quantity is >0,

$a^2d^2-2abcd+b^2c^2$ is >0,

$\therefore\ a^2d^2+b^2c^2$ is $>2abcd$,

and $a^2d^2+b^2c^2+a^2c^2+b^2d^2>a^2c^2+2abcd+b^2d^2$,

or $(a^2+b^2)\,(c^2+d^2)>(ac+bd)^2$,

and $\therefore\ \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}>ac+bd$.

20. Except when $a=b=c$,

$a^2>a^2-(b-c)^2>(a+b-c)\,(a-b+c)$,

$b^2>b^2-(a-c)^2>(b+a-c)\,(b-a+c)$

$c^2>c^2-(a-b)^2>(c+a-b)\,(c+b-a)^2$,

$\therefore\ a^2b^2c^2>(a+b-c)^2\,(a-b+c)^2\,(b+c-a)^2$,

and $abc>(a+b-c)\,(a-b+c)\,(b+c-a)$.

21. Except when $a=1$,

$(a-1)^2$ is >0,

and $\therefore\ a^2-a+1>a$,

$(a^2-a+1)\,(a+1)>a\,(a+1)$,

or $a^3+1>a^2+a$.

22. Let a be $>b$, then

$ma=ma$ and is $>mb$,

$nb<na$ and $=nb$,

$\therefore\ ma+nb<(m+n)\,a$ and $>(m+n)\,b$,

or $\dfrac{ma+nb}{m+n}<a$ and $>b$.

23. Substituting the value of x in the given expression, we find that

$$\frac{x+2a}{x-2a}=\frac{3b+a}{b-a} \text{ and } \frac{x+2b}{x-2b}=\frac{3a+b}{a-b};$$

and the sum of these two fractions = 2.

24. Extract the square root of the given expression by the ordinary process; the remainder is found to be

$$\left\{r-\frac{p}{2}\left(q-\frac{p^2}{4}\right)\right\}x+s-\frac{1}{4}\left(q-\frac{p^2}{4}\right)^2.$$

In order that the proposed expression may be a perfect square, this remainder must vanish,

$$\therefore\ r=\frac{p}{2}\left(q-\frac{p^2}{4}\right), \text{ and } s=\frac{1}{4}\left(q-\frac{p^2}{4}\right)^2.$$

Hence $q=2\sqrt{s}+\frac{p^2}{4}$, and $r=p\sqrt{s}$ or $p^2s=r^2$.

25. Extract the cube root by the ordinary process; the remainder is found to be

$$\left(c-\frac{b^2}{3a}\right)x+d-\frac{b^3}{27a^2}.$$

In order therefore that the given expression may be a perfect cube, we must have

$$c=\frac{b^2}{3a}, \text{ and } d=\frac{b^3}{27a^2},$$

cubing the first of these equations and dividing the result by the second, we find that $ac^3=db^3$.

26. No: for multiplying the *first* equation by 2, and adding the result to the *second*, we obtain the *third* equation, which shews that the three equations are not independent.

28. By Art. 70, we have

$$x_1+x_2=-\frac{b}{a}, \text{ and } x_1x_2=\frac{c}{a},$$

$$\therefore \frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{x_1^2 + x_2^2}{x_1 x_2} = \frac{(x_1 + x_2)^2}{x_1 x_2} - 2$$

$$= \frac{b^2}{ac} - 2 = \frac{b^2 - 2ac}{ac}.$$

29. $$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}.$$

The quantity $\left(x + \frac{a}{2}\right)^2$ cannot be negative, if therefore b be $> \frac{a^2}{4}$, or a^2 be $< 4b$, $x^2 + ax + b$ must be positive.

30. Eliminating x from the two equations, we find

$$x^2 = \frac{bc' - b'c}{ab' - a'b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

eliminating x^2, we find

$$x = \frac{ac' - a'c}{a'b - ab'},$$

and $$\therefore x^2 = \left(\frac{ac' - a'c}{a'b - ab'}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Equating these two values of x^2 and reducing, we get the given equation of condition.

31. From the first and second equations, we find

$$(ab' - a'b)\, x = b'c - bc', \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and from the first and third,

$$(ab'' - a''b)\, x = b''c - bc'', \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

dividing (i) by (ii) and reducing, we get the given equation of condition.

32. Since the square of a possible quantity is always positive, we have,

$$(\sqrt{a} - \sqrt{c})^2 > 0,$$

$$\text{or } a + c > 2\sqrt{ac},$$

à fortiori, $\therefore\ 2(a + c) > 2\sqrt{ac},$

and $\therefore\ 2b(a + c) > 2ac,$ since $b = \sqrt{ac},$

$$\therefore\ a^2 + b^2 + c^2 + 2b(a + c) > a^2 + b^2 + c^2 + 2ac,$$

$$a^2 + b^2 + c^2 > a^2 + b^2 + c^2 - 2ab + 2ac - 2bc,$$

$$\text{or } a^2 + b^2 + c^2 > (a - b + c)^2.$$

33. $(a - a')^2$ is > 0, or $a^2 - 2aa' + a'^2 > 0,$

$$\therefore\ a^2 + a'^2 > 2aa'.$$

$$\text{So } b^2 + b'^2 > 2bb',$$

$$c^2 + c'^2 > 2cc',$$

$$\therefore\ a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 > 2(aa' + bb' + cc')$$

$$\text{or } 2 > 2(aa' + bb' + cc')$$

$$\text{or } aa' + bb' + cc' < 1.$$

34. ${}_nP_r + {}_{n+1}P_{r-1}$

$$= \frac{n(n+1)\ldots(n+r-1)}{1.2\ldots r}$$

$$+ \frac{(n+1)(n+2)\ldots(n+r-1)}{1.2\ldots(r-1)}$$

$$= \frac{(n+1)(n+2)\ldots(n+r-1)}{1.2\ldots r}(n+r)$$

$$= \frac{(n+1)(n+2)\ldots\{(n+1)+r-1\}}{1.2\ldots r}$$

$$= {}_{n+1}P_r.$$

TRIGONOMETRY.

GENERAL QUESTIONS.

1. SINCE we may subtract $2n\,180^0$ from any angle without altering its trigonometrical ratios (Art. 18), we have

$$\sin\{(2n+1)\,180^0+\theta\} = \sin(180^0+\theta) = -\sin\theta,$$

$$\cos\{(2n+1)\,180^0+\theta\} = \cos(180^0+\theta) = -\cos\theta.$$

2. While θ changes from 0 to 90^0, $\sin\theta$ and $\cos\theta$ are both positive, and therefore their sum is positive: while θ changes from 90^0 to 135^0, $\sin\theta$ is positive and $\cos\theta$ is negative, but $\sin\theta$ is $>\cos\theta$; therefore $\sin\theta+\cos\theta$ is positive: while θ changes from 135^0 to 180, $\sin\theta$ is positive and $\cos\theta$ negative, but $\cos\theta$ is $>\sin\theta$; $\therefore$ $\sin\theta+\cos\theta$ is negative: similarly, while θ changes from 180^0 to 315^0, $\sin\theta+\cos\theta$ is negative, and while θ changes from 315^0 to 360^0, $\sin\theta+\cos\theta$ is positive.

3. If θ be between 90^0 and 135^0, $\sin 2\theta$ and $\cos 2\theta$ are negative, and $\tan 2\theta$ is positive; if θ be between 135^0 and 180^0 $\sin 2\theta$ and $\tan 2\theta$ are negative, and $\cos 2\theta$ is positive.

4. Ans. $n\,180^0-\theta$, n being any integer.

5. ... $n\,360^0$, n being any integer.

6. See Art. 12.

7. $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$, $\cos A = \dfrac{1}{\sqrt{1+\tan^2 A}}$,

$\cot A = \dfrac{1}{\tan A}$, $\sec A = \sqrt{1+\tan^2 A}$, $\operatorname{cosec} A = \dfrac{\sqrt{1+\tan^2 A}}{\tan A}$.

8. $\cos A = 1 - \text{vers}\, A$, $\sin A = \sqrt{2\,\text{vers}\, A - \text{vers}^2 A}$,

$$\tan A = \frac{\sqrt{2\,\text{vers}\, A - \text{vers}^2 A}}{1 - \text{vers}\, A};$$

$\sec A$, $\text{cosec}\, A$, and $\cot A$ are the reciprocals of these.

9. $\cos A = \text{suvers}\, A - 1$, $\sin A = \sqrt{2\,\text{suvers}\, A - \text{suvers}^2 A}$,

$$\tan A = \frac{\sqrt{2\,\text{suvers}\, A - \text{suvers}^2 A}}{\text{suvers}\, A - 1}.$$

10. Replacing $\cot\theta$ by $\frac{1}{\tan\theta}$, and solving the resulting quadratic with respect to $\tan\theta$, we find

$$\tan\theta = \tfrac{1}{2}(m \pm \sqrt{m^2 - 4}).$$

Whence, by Ex. 7,

$$\sin\theta = \sqrt{\frac{m \pm \sqrt{m^2 - 4}}{2m}}, \text{ and } \cos\theta = \sqrt{\frac{m \mp \sqrt{m^2 - 4}}{2m}}.$$

11. Replacing $\cos\theta$ by $\sqrt{1 - \sin^2\theta}$, and solving the resulting equation, we obtain

$$\sin\theta = \pm\sqrt{\frac{1 \pm \sqrt{1 - 4m^2}}{2}}.$$

12. The ratio of the numerical representatives of two angles will be the same, whatever be the unit in terms of which they are expressed. Let the angle, which is expressed by $36\frac{7}{60}$ when $\frac{1}{90}$th part of a right angle is the unit, be expressed by x when $\frac{1}{100}$th part of a right angle is the unit; then

$$\frac{x}{100} = \frac{\text{the given angle}}{\text{a right angle}} = \frac{36\frac{7}{60}}{90},$$

whence $x = 40\frac{7}{54}$.

13. Let x be the number of degrees, then, as in the last example,

$$\frac{x}{40} = \frac{90}{30}, \text{ whence } x = 120.$$

14. See the last observation in Art. 16.

FORMULÆ INVOLVING FUNCTIONS OF ONE ANGLE.

1. The *first six* equations may be proved by Arts. 14 and 18.

Since $\sin 180^0 = \tan 180^0 = 0$, and $\cos 180^0 = \sec 180^0 = -1$, we have

$$\sin\theta + \cos\theta + \tan\theta - \sec\theta = 0, \text{ when } \theta = 180^0.$$

Again, since $\sin\theta \cot\theta = \cos\theta$, and $\cos 0 = 1$, we have

$$\sin\theta \cot\theta = 1, \text{ when } \theta = 0.$$

Also, since $\sin 45^0 = \cos 45^0$, we have

$$\sin\theta - \cos\theta = 0, \text{ when } \theta = 45^0.$$

Lastly, since $\sin 135^0 = \frac{1}{\sqrt{2}}$, and $\cos 135^0 = -\frac{1}{\sqrt{2}}$,

we have $\sin\theta + \cos\theta = 0$, when $\theta = 135^0$.

2. By Ex. 7, p. 78, we find

$$\frac{1}{\cos\theta} = \frac{1}{a^{\frac{1}{3}}}\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}, \quad \frac{1}{\sin\theta} = \frac{1}{b^{\frac{1}{3}}}\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}};$$

$$\text{whence } \frac{a}{\cos\theta} + \frac{b}{\cos\theta} = a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} + b^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}$$

$$= (a^{\frac{2}{3}} + b^{\frac{2}{3}})\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

3. $\text{Sin}(\alpha - \theta) = \cos(\alpha + \theta) = \sin(90^0 - \alpha - \theta)$;

$$\therefore \alpha - \theta = 90^0 - \alpha - \theta, \quad \text{or } \alpha = 45^0,$$

θ is perfectly arbitrary.

4. The given equations are equivalent to

$$m = \frac{1}{\sin\theta} - \sin\theta = \frac{1 - \sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta},$$

$$\text{and } n = \frac{1}{\cos\theta} - \cos\theta = \frac{1 - \cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}.$$

Hence $m^2n = \cos^3\theta$, and $mn^2 = \sin^3\theta$;

and therefore $m^{\frac{1}{3}}n^{\frac{2}{3}} = \cos^2\theta$, and $m^{\frac{2}{3}}n^{\frac{1}{3}} = \cos^2\theta$.

Or, adding, $m^{\frac{1}{3}}n^{\frac{2}{3}} + m^{\frac{2}{3}}n^{\frac{1}{3}} = \sin^2\theta + \cos^2\theta$;

i. e. $m^{\frac{1}{3}}n^{\frac{1}{3}}(m^{\frac{1}{3}} + n^{\frac{1}{3}}) = 1$.

5. Ans. $\theta = (2n + 1)\,180^0$, n being any integer.

6. $$\frac{\tan^2\alpha}{\tan^2\gamma} = \frac{\tan^2\theta}{\tan^2\phi} = \frac{\sec^2\theta - 1}{\sec^2\phi - 1} = \frac{\cos\beta - \cos\alpha}{\cos\beta - \cos\gamma}\cdot\frac{\cos\gamma}{\cos\alpha};$$

whence $$\cos\beta = \frac{\sin^2\gamma\cos^2\alpha - \cos^2\gamma\sin^2\alpha}{\sin^2\gamma\cos\alpha - \cos\gamma\sin^2\alpha}.$$

This may be reduced to

$$\tan\frac{\beta}{2} = \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}.$$

7. $a.\sin^2\theta + b\cos^2\theta = m(\cos^2\theta + \sin^2\theta)$;

$\therefore (a - m).\tan^2\theta = m - b$, and so $(b - n).\tan^2\phi = n - a$;

$$\therefore \frac{b^2}{a^2} = \frac{\tan^2\theta}{\tan^2\phi} = \frac{(m - b).(b - n)}{(a - m).(n - a)};$$

whence, $$\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}.$$

8. $$a.\cos\theta + b\sin\theta = \frac{a + b\tan\theta}{\sqrt{1 + \tan^2\theta}} = \sqrt{a^2 + b^2}.$$

FORMULÆ INVOLVING MORE THAN ONE ANGLE.

1. $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}.$$

$$\cot\theta - \tan\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta.\cos\theta} = 2\frac{\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta.$$

$$\sqrt{1+\sin 2\theta} = \sqrt{\sin^2\theta + \cos^2\theta + 2\sin\theta.\cos\theta} = \sin\theta + \cos\theta.$$

$$\operatorname{cosec} 2\theta + \cot 2\theta = \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta.\cos\theta} = \cot\theta.$$

$$\cot\theta + \tan\theta = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta.\cos\theta} = \frac{2}{\sin 2\theta} = 2\operatorname{cosec}\theta.$$

$$\cot\theta - \tan\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta.\cos\theta} = 2\frac{\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta.$$

$$2\operatorname{cosec} 2\theta = \frac{2}{2\sin\theta.\cos\theta} = \frac{1}{\cos\theta}.\frac{1}{\sin\theta} = \sec\theta.\operatorname{cosec}\theta.$$

$$\sec(\theta \pm \phi) = \frac{1}{\cos\theta.\cos\phi\,(1 \mp \tan\theta.\tan\phi)} = \frac{\sec\theta.\sec\phi}{1 \mp \tan\theta.\tan\phi}.$$

$$\cos\theta = \frac{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta} = \frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}.$$

$$\begin{aligned}\sin 3\theta &= \sin 2\theta.\cos\theta + \cos 2\theta.\sin\theta \\ &= 2\sin\theta.(1-\sin^2\theta) + \sin\theta\,(1-2\sin^2\theta) \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

$$\tan(45^0+\theta) = \frac{1+\tan\theta}{1-\tan\theta} = \sqrt{\frac{1+2\tan\theta+\tan^2\theta}{1-2\tan\theta+\tan^2\theta}} = \sqrt{\frac{1+\sin 2\theta}{1-\sin 2\theta}}.$$

$$\begin{aligned}\tan^2\theta - \tan^2\phi &= \frac{(\sin\theta.\cos\phi+\cos\theta.\sin\phi)(\sin\theta\cos\phi-\cos\theta\sin\phi)}{\cos^2\theta\cos^2\phi} \\ &= \frac{\sin(\theta+\phi)\sin(\theta-\phi)}{\cos^2\theta\cos^2\phi}.\end{aligned}$$

$$\frac{1+\sin\theta}{1+\cos\theta} = \frac{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}{2\cos^2\frac{\theta}{2}} = \frac{1}{2}\left(1+\tan\frac{\theta}{2}\right)^2.$$

$$\frac{\sin 3\theta + \cos 3\theta}{\sin 3\theta - \cos 3\theta} = \frac{\sin 2\theta\,(\cos\theta - \sin\theta) + \cos 2\theta\,(\cos\theta + \sin\theta)}{\sin 2\theta\,(\cos\theta + \sin\theta) - \cos 2\theta\,(\cos\theta - \sin\theta)}$$

$$= \frac{\{\sin 2\theta + (\cos\theta + \sin\theta)^2\}(\cos\theta - \sin\theta)}{\{\sin 2\theta - (\cos\theta - \sin\theta)^2\}(\cos\theta + \sin\theta)}$$

$$= \frac{2\sin 2\theta + 1}{2\sin 2\theta - 1}\tan(45^0 - \theta).$$

$$\sin n\theta + \sin(n-2)\theta = 2\sin\tfrac{1}{2}\{n + (n-2)\}\theta.\cos\tfrac{1}{2}\{n - (n-2)\}\theta$$

$$= 2\sin(n-1)\theta.\cos\theta. \quad \text{(Art. 23.)}$$

$$\sin^2\theta - \sin^2\phi = \tfrac{1}{2}\{(1 - \cos 2\theta) - (1 - \cos 2\phi)\}$$

$$= \tfrac{1}{2}(\cos 2\phi - \cos 2\theta) = \sin(\theta + \phi).\sin(\theta - \phi).$$

$$\sec^2\frac{\theta}{2} = \frac{2}{2\cos^2\dfrac{\theta}{2}} = \frac{2}{1 + \cos\theta} = \frac{2\sec\theta}{1 + \sec\theta}.$$

$$\frac{\operatorname{vers}\theta}{\operatorname{vers}(180^0 - \theta)} = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{2\sin^2\dfrac{\theta}{2}}{2\cos^2\dfrac{\theta}{2}} = \tan^2\frac{\theta}{2}.$$

$$\frac{1 - 2\sin^2\theta}{1 + \sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{(\cos\theta + \sin\theta)^2} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1 - \tan\theta}{1 + \tan\theta}.$$

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{(\cos\theta + \sin\theta)^2}{\cos^2\theta - \sin^2\theta} = \frac{\sin 2\theta + 1}{\cos 2\theta} = \tan 2\theta + \sec 2\theta.$$

$$\cos^2(\theta + \phi) - \sin^2\theta = \tfrac{1}{2}[\{1 + \cos 2(\theta + \phi)\} - (1 - \cos 2\theta)]$$

$$= \tfrac{1}{2}\{\cos 2(\theta + \phi) + \cos 2\theta\} = \cos(2\theta + \phi).\cos\phi.$$

$$\frac{2\operatorname{cosec} 2\theta - \sec\theta}{2\operatorname{cosec} 2\theta + \sec\theta} = \frac{1 - \sin\theta}{1 + \sin\theta} = \frac{\left(\cos\dfrac{\theta}{2} - \sin\dfrac{\theta}{2}\right)^2}{\left(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right)^2} = \cot^2\left(45^0 + \frac{\theta}{2}\right).$$

$$\sec\theta = \frac{2}{\sqrt{4\cos^2\theta}} = \frac{2}{\sqrt{2+2\cos 2\theta}} = \frac{2}{\sqrt{2+\sqrt{2+2\cos 4\theta}}},$$

$$\text{for } 2\cos 2\theta = \sqrt{2\,(1+\cos 4\theta)}.$$

$$(\sin\theta+\sin\phi).(\sin\theta-\sin\phi)$$
$$= 2\sin\tfrac{1}{2}(\theta+\phi).\cos\tfrac{1}{2}(\theta-\phi)\times 2\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)$$
$$= \sin(\theta+\phi).\sin(\theta-\phi).$$

$$\frac{\sin\theta-\sin\phi}{\cos\theta-\cos\phi} = \frac{\cos\frac{1}{2}(\theta+\phi).\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta+\phi).\sin\frac{1}{2}(\phi-\theta)} = -\cot\tfrac{1}{2}(\theta+\phi).$$

$$\tan\theta+\cot\theta = \frac{\sin^2\theta+\cos^2\theta}{\cos\theta.\sin\theta} = \frac{2}{\sin 2\theta} = 2\operatorname{cosec} 2\theta.$$

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan^2\tfrac{1}{2}\theta.$$

$$\frac{\cos\phi+\cos\theta}{\cos\phi-\cos\theta} = \frac{\cos\frac{1}{2}(\theta+\phi).\cos\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta+\phi).\sin\frac{1}{2}(\theta-\phi)} = \cot\frac{\theta+\phi}{2}\cot\frac{\theta-\phi}{2}.$$

$$2\,(\sin^2\theta.\sin^2\phi+\cos^2\theta.\cos^2\phi)$$
$$= \tfrac{1}{2}\{(1-\cos 2\theta).(1-\cos 2\phi)+(1+\cos 2\theta).(1+\cos 2\phi)\}$$
$$= 1+\cos 2\theta.\cos 2\phi.$$

$$\cos(A+B).\sin(A-B)+\cos(B+C)\sin(B-C)$$
$$+\cos(C+D)\sin(C-D)+\cos(D+A).\sin(D-A)$$
$$= \tfrac{1}{2}\{(\sin 2A-\sin 2B)+(\sin 2B-\sin 2C)$$
$$+(\sin 2C-\sin 2D)+(\sin 2D-\sin 2A)\} = 0.$$

$$\frac{2\sin\theta-\sin 2\theta}{2\sin\theta+\sin 2\theta} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan^2\frac{\theta}{2}.$$

$$2\operatorname{vers}\left(90^0+\frac{\theta}{2}\right).\operatorname{vers}\left(90^0-\frac{\theta}{2}\right) = 2\left(1+\sin\frac{\theta}{2}\right).\left(1-\sin\frac{\theta}{2}\right)$$
$$= 2\cos^2\frac{\theta}{2} = 1+\cos\theta = \operatorname{vers}(180^0+\theta).$$

$$\sin(\theta+\phi)+\sin(\theta-\phi)=2\sin\theta\cos\phi=2\sin\theta\left(1-2\sin^2\frac{\phi}{2}\right);$$

$$\therefore\ \sin(\theta+\phi)-\sin\theta=\sin\theta-\sin(\theta-\phi)-4\sin\theta.\sin^2\frac{\phi}{2}.$$

$$4\sin(\theta-\phi).\sin(m\theta-\phi).\cos(\theta-m\theta)$$
$$=2\left[\cos(1-m)\theta-\cos\left\{(1+m)\theta-2\phi\right\}\right]\cos(1-m)\theta$$
$$=1+\cos(2-2m)\theta-\cos(2\theta-2\phi)-\cos(2m\theta-2\phi).$$

2. Since $\dfrac{n}{m}=\dfrac{\tan(\alpha-\theta).\cos^2(\alpha-\theta)}{\tan\theta.\cos^2\theta}=\dfrac{\sin(2\alpha-2\theta)}{\sin 2\theta}$;

$$\therefore\ \frac{n-m}{n+m}=\frac{\sin(2\alpha-2\theta)-\sin 2\theta}{\sin(2\alpha-2\theta)+\sin 2\theta}$$
$$=\frac{\sin(\alpha-2\theta).\cos\alpha}{\cos(\alpha-2\theta).\sin\alpha}=\frac{\tan(\alpha-2\theta)}{\tan\alpha}.$$

3. Since

$$m.\sin\beta.(\cos\alpha+\sin\alpha.\tan\theta)=n\sin\alpha(\cos\beta-\sin\beta.\tan\theta);$$

$$\therefore\ \tan\theta=\frac{n\tan\alpha-m\tan\beta}{(m+n)\tan\alpha.\tan\beta}.$$

4. Since $\tan\theta=2\sin\theta.\cos\theta$;

$\therefore\ \sin\theta=0=\sin n\pi$, whence chd. $\theta=0$ or ± 2;

or, $1=2\cos^2\theta$, and $\cos\theta=\pm\dfrac{1}{\sqrt{2}}$;

whence chd. $\theta=2\sin\dfrac{\theta}{2}=\sqrt{2}\sqrt{1-\cos\theta}=\sqrt{2\mp\sqrt{2}}$.

5. $2B=A+C$, and therefore

$$\frac{\sin A-\sin C}{\cos C-\cos A}=\frac{\cos\dfrac{A+C}{2}\sin\dfrac{A-C}{2}}{\sin\dfrac{A+C}{2}\sin\dfrac{A-C}{2}}=\frac{\cos B}{\sin B}.$$

6. $$\text{Tan}\,\frac{\alpha+\beta}{2} = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$= \frac{\tan\frac{\beta}{2}\left(1+\tan^2\frac{\beta}{2}\right)}{1-\tan^4\frac{\beta}{2}} = \tfrac{1}{2}\tan\beta = \tan\gamma,$$

$$\text{i.e. } \alpha+\beta = 2\gamma.$$

7. Since $m^2 = 1+3\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)^2 = 4\,\dfrac{1-\tan^{\frac{2}{3}}\theta+\tan^{\frac{4}{3}}\theta}{(1+\tan^{\frac{2}{3}}\theta)^2}$

$$= \frac{4\,(1+\tan^2\theta)}{(1+\tan^{\frac{2}{3}}\theta)^3} = \frac{4}{(\cos^{\frac{2}{3}}\theta+\sin^{\frac{2}{3}}\theta)^3};$$

$$\therefore \left(\frac{2}{m}\right)^{\frac{2}{3}} = \cos^{\frac{2}{3}}\theta+\sin^{\frac{2}{3}}\theta.$$

8. By division $\cos\theta = \dfrac{m}{n}\cos\phi$; also $\sin\theta = m\sin\phi$;

$$\therefore 1 = m^2(1-\cos^2\phi)+\frac{m^2}{n^2}\cos^2\phi,$$

$$\text{or } n^2(1-m^2) = m^2(1-n^2)\cos^2\phi;$$

$$\therefore \cos\phi = \pm\frac{n}{m}\cdot\sqrt{\frac{1-m^2}{1-n^2}};$$

and $\therefore \cos\theta = \pm\sqrt{\dfrac{1-m^2}{1-n^2}}$, and $\sin\theta = \pm\sqrt{\dfrac{m^2-n^2}{1-n^2}}$.

9. $\text{Cos}\,\alpha+\sin\alpha\,.\cot\theta = m$, and therefore

$$\cot\theta = \frac{m}{\sin\alpha}-\cot\alpha = \cot\phi-\cot\alpha \text{ suppose,}$$

(for $\cot\phi$ may have any magnitude),

$$\text{or } \cot\theta = \frac{\sin(\alpha-\phi)}{\sin\phi\sin\alpha};$$

$$\therefore L\cot\theta = L\sin(\alpha-\phi) - L\sin\phi - L\sin\alpha + 20,$$

where ϕ is known from

$$L\cot\phi = \log_{10} m - L\sin\alpha + 20.$$

10. $$\text{Tan}\,\phi = \frac{2\sin\theta.\sin\psi}{\sin\theta\cos\psi + \cos\theta.\sin\psi} = \frac{2}{\cot\psi + \cot\theta},$$

$$\therefore \frac{1}{\tan\theta} + \frac{1}{\tan\psi} = \frac{2}{\tan\phi}.$$

11. Since $\tan(\theta+\phi+\psi) = \tan(2n+1)90^\circ$;

$$\therefore \frac{\tan\theta + \tan\phi + \tan\psi - 3\tan\theta\tan\phi\tan\psi}{1-(\tan\phi\tan\psi + \tan\theta\tan\psi + \tan\theta\tan\phi)} = \infty;$$

$$\therefore 1-(\tan\phi\tan\psi + \tan\theta\tan\psi + \tan\theta\tan\phi) = 0.$$

Also $\sin 2\theta + \sin 2\phi + \sin 2\psi$

$$= 2\sin(\theta+\phi)\cos(\theta-\phi) \pm 2\cos(\theta+\phi).\cos\psi;$$

$$= \pm 2\cos\psi\{\cos(\theta+\phi) + \cos(\theta-\phi)\}$$

$\{$since $\sin(\theta+\phi) = \pm\cos\psi$, and $\cos(\theta+\phi) = \pm\sin\psi\}$

$$= \pm 4\cos\theta.\cos\phi.\cos\psi.$$

12. $$\text{Tangent of sum} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = \frac{5}{5} = 1.$$

13. Since $$\tan\frac{\theta}{2} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1-\cos\theta}{\sin\theta};$$

$$\therefore \frac{1-\cos\theta}{\sin\theta} = \frac{1-\sin^2\theta}{\sin\theta};$$

$$\therefore \cos^2\theta + \cos\theta = 1, \quad \text{or} \quad \cos\theta = \tfrac{1}{2}(\pm\sqrt{5} - 1),$$

which determines θ.

14. Since $n = \dfrac{\sin(a-x)\cos x}{\cos(a-x).\sin x}$;

$$\therefore \frac{n-1}{n+1} = \frac{\sin(a-2x)}{\sin\{(a-x)+x\}} = \frac{\sin(a-2x)}{\sin a}.$$

15. $\operatorname{Cos}(4\theta+\theta)$

$= \cos\theta.(2\cos^2 2\theta - 1) - \sin\theta.2\sin 2\theta\cos 2\theta$

$= \cos\theta\{2(2\cos^2\theta - 1)^2 - 1 + 4(\cos^2\theta - 1)(2\cos^2\theta - 1)\}$

$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

16. $$\operatorname{Tan} 3\theta = \frac{\tan\theta + \dfrac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta.\dfrac{2\tan\theta}{1-\tan^2\theta}} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

17. $\operatorname{Sin} a = \sin(\theta+a) - \sin(\theta-a) - \{\sin(2\theta+a) - \sin(2\theta-a)\}$

$= 2\sin a(\cos\theta - \cos 2\theta);$

$\therefore 4\cos^2\theta - 2\cos\theta = 1;$

$$\text{and } \therefore \cos\theta = \frac{\pm\sqrt{5}+1}{4},$$

$$\text{and } \theta = 2n\pi \pm \frac{\pi}{5}, \text{ or } (2n+1)\pi \pm \frac{\pi}{10}.$$

18. $2 = 2\sin^2 3\theta + 4\sin^2 3\theta(1 - \sin^2 3\theta) = 6\sin^2 3\theta - 4\sin^4 3\theta;$

$$\therefore (\sin^2 3\theta - \tfrac{3}{4})^2 = \tfrac{9}{16} - \tfrac{1}{2} = \tfrac{1}{16},$$

$$\sin 3\theta = \pm\sqrt{\frac{3\pm 1}{4}} = \pm 1 \quad \text{or} \quad \pm\frac{1}{\sqrt{2}};$$

$$\therefore\ \theta = \tfrac{1}{3}\left(n\pi \pm \frac{\pi}{2}\right) \text{ or } \tfrac{1}{3}\left(n\pi \pm \frac{\pi}{4}\right),$$

$$= \frac{2n \pm 1}{6}\pi \text{ or } \frac{4n \pm 1}{12}\pi.$$

19. The second side $= \sin\theta \cdot \dfrac{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}}{\cos\theta \cdot \cos\frac{\theta}{2}}$

$$= \sin\theta \cdot \frac{\cos\left(\theta - \frac{\theta}{2}\right)}{\cos\theta \cdot \cos\frac{\theta}{2}} = \tan\theta.$$

Hence $\cot 2\theta = 0 = \cot(2n+1)\,90^0$;

$$\therefore\ \theta = (2n+1)\,45^0.$$

20. $\text{Cos}\theta\cos\psi + \sin\theta\sin\psi$

$$= \cos(\theta+\psi) + \sin(\theta+\psi)\tan\phi;$$

$$\therefore\ \tan\theta \cdot \tan\psi = -\tan\theta\tan\psi + \tan\theta\tan\phi + \tan\phi\tan\psi,$$

$$\text{or } \frac{2}{\tan\phi} = \frac{1}{\tan\theta} + \frac{1}{\tan\psi}.$$

21. If $1 = \dfrac{\sin^2\phi}{\sin^2\theta} + \cos^2\phi\cos^2\psi$, then $\sin\psi = \dfrac{\tan\phi}{\tan\theta}$.

For $\sin^2\psi = 1 - \left(\dfrac{1}{\cos^2\phi} - \dfrac{\tan^2\phi}{\sin^2\theta}\right) = \dfrac{\tan^2\phi}{\sin^2\theta} - \tan^2\phi = \dfrac{\tan^2\phi}{\tan^2\theta}$.

22. $\dfrac{\cos\theta}{\cos\phi} = \cos\psi$; $\therefore\ \dfrac{\cos\phi - \cos\theta}{\cos\phi + \cos\theta} = \dfrac{1-\cos\psi}{1+\cos\psi}$,

$$\text{or } \tan\frac{\theta+\phi}{2} \cdot \tan\frac{\theta-\phi}{2} = \tan^2\frac{\psi}{2}.$$

23. For $\dfrac{1+m}{1-m} = \dfrac{\sin(2\theta+\phi) + \sin\phi}{\sin(2\theta+\phi) - \sin\phi} = \dfrac{\tan(\theta+\phi)}{\tan\theta}$.

24. $\text{Cos}\,2\theta + 2\cos 2\theta \,.\, \cos\theta = 0$;

$$\therefore\ \cos 2\theta = 0, \text{ and } 1 + 2\cos\theta = 0,$$

whence $\theta = (2n+1)\frac{\pi}{4}$, and $\theta = (2n+1)\pi \pm \frac{\pi}{3}$.

NUMERICAL VALUES OF FUNCTIONS.

1. $\text{Sin}\,\theta + \sin(72^0 + \theta) - \sin(72^0 - \theta)$

$$- \{\sin(36^0 + \theta) - \sin(36^0 - \theta)\}$$

$$= \sin\theta \,.\, \{1 + 2(2\cos^2 36^0 - 1) - 2\cos 36^0\}$$

$$= \sin\theta \,.\, \left\{-1 + \tfrac{1}{4}(6 + 2\sqrt{5}) - \frac{1+\sqrt{5}}{2}\right\}$$

$$= 0.$$

2. $\text{Sin}\,(45^0 - 30^0) = \frac{1}{\sqrt{2}}(\cos 30^0 - \sin 30^0) = \frac{\sqrt{3}-1}{2\sqrt{2}}$;

$$\therefore\ \cos 15^0 = \sqrt{1 - \sin^2 15^0} = \sqrt{1 - \frac{4 - 2\sqrt{3}}{8}} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

$$\text{chd}\,15^0 = 2\sin\frac{15^0}{2} = \sqrt{2}\sqrt{1 - \cos 15^0} = \sqrt{2 - \frac{\sqrt{3}+1}{\sqrt{2}}},$$

$$\tan 15^0 = \frac{\sin 15^0}{\cos 15^0} = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

3. $\text{Sin}\,75^0 = \cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$ by (2).

4. $\text{Sin}\,18^0 \,.\, \sin 54^0 = \frac{\sqrt{5}-1}{4} \,.\, \frac{\sqrt{5}+1}{4} = \frac{1}{4} = \sin^2 30^0$.

5. $2\cos 11^0\,15' = \sqrt{2}\sqrt{1 + \cos 22^0\,30'}$

$$= \sqrt{2 + \sqrt{2}\sqrt{1 + \cos 45^0}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

6. $\text{Tan}\, 2\phi = \dfrac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{6}{8}$;

$\therefore \tan(\theta + 2\phi) = \dfrac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \cdot \frac{3}{4}} = \frac{25}{25} = \tan 45^0$; $\therefore \theta + 2\phi = 45^0$.

7. $\text{Sin}(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{\sqrt{1 + \tan^2\theta}\sqrt{1 + \tan^2\phi}}$

$$= \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{15}}}{\sqrt{1 + \frac{1}{3}} \cdot \sqrt{1 + \frac{1}{15}}} = \frac{\sqrt{3}}{2 \times 4} \cdot (\sqrt{5} + 1)$$

$$= \cos 60^0 \cdot \cos 36^0.$$

8. $\text{Chd}\, 120^0 = 2 \sin 60^0 = \sqrt{3} = \tan 60^0$.

9. $\text{Tan}(45^0 + \phi) = \dfrac{1 + \frac{1}{239}}{1 - \frac{1}{239}} = \frac{120}{119}$,

$$\tan 2\theta = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12};$$

$$\therefore \tan 4\theta = \frac{\frac{10}{12}}{1 - \frac{25}{144}} = \frac{120}{119};$$

$\therefore 4\theta = 45^0 + \phi$, or $4\theta - \phi = 45^0$.

10. $\text{Sin}\, 12^0 = \sin(30^0 - 18^0)$

$$= \tfrac{1}{8}\{\sqrt{10 + 2\sqrt{5}} - \sqrt{3}(\sqrt{5} - 1)\},$$

$$\cos 22^0\tfrac{1}{2} = \sqrt{\frac{1 + \cos 45^0}{2}} = \tfrac{1}{2}\sqrt{2 + \sqrt{2}}.$$

$$\cos 9^0 = \tfrac{1}{2}(\sqrt{1 + \sin 18^0} + \sqrt{1 - \sin 18^0})$$

$$= \tfrac{1}{4}(\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}).$$

SOLUTION OF, AND FORMULÆ RELATING TO, TRIANGLES.

1. $\text{Cot } A + \cot B = \dfrac{AD + BD}{CD}$, fig. (1),

$$= \frac{c}{a \sin B} = \frac{c}{a} \operatorname{cosec} B.$$

$$\text{The first side} = \sin \frac{A}{2} \left(\cos \frac{B+C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$+ \sin \frac{B}{2} \left(\cos \frac{A+C}{2} + \sin \frac{A}{2} \sin \frac{C}{2} \right) + \sin^2 \frac{C}{2}$$

$$= \sin^2 \frac{C}{2} + \cos \frac{C}{2} \cdot \sin \frac{A+B}{2}$$

$$= \sin^2 \frac{C}{2} + \cos^2 \frac{C}{2} = 1.$$

The second side

$$= (a^2 + b^2) \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right)$$

$$= a^2 + b^2 - 2ab \cos C = c^2.$$

$$\text{The first side} = \sin A + 2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}$$

$$= 2 \cos \frac{A}{2} \left(\cos \frac{B+C}{2} + \cos \frac{B-C}{2} \right)$$

$$= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

$$\text{The first side} = \sin 2A + 2 \sin (B + C) \cos (B - C)$$

$$= 2 \sin A \{ \cos (B - C) - \cos (B + C) \} = 4 \sin A \sin B \sin C.$$

$$\text{Cot}\frac{A}{2} = \tan\frac{B+C}{2} = \frac{\tan\frac{B}{2} + \tan\frac{C}{2}}{1 - \tan\frac{B}{2}\tan\frac{C}{2}}$$

$$= \frac{\cot\frac{C}{2} + \cot\frac{B}{2}}{\cot\frac{B}{2}\cot\frac{C}{2} - 1};$$

$$\therefore \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}.$$

$$\frac{a^2 - b^2}{c^2} = \frac{a-b}{c}.\frac{a+b}{c} = \frac{\sin A - \sin B}{\sin C}.\frac{\sin A + \sin B}{\sin C}$$

$$= \frac{\sin(A+B).\sin(A-B)}{\sin^2 C} = \frac{\sin(A-B)}{\sin C}.$$

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{\cos\frac{A+B}{2}\ \cos\frac{A-B}{2}}{\sin^2\frac{C}{2}} = \frac{\cos A + \cos B}{2\sin^2\frac{C}{2}}.$$

The first member

$= (a\cos B + b\cos A) + (b\cos C + c\cos B) + (c\cos A + a\cos C)$

$= c + a + b$. See fig. (1).

2. $r.(a+b+c) = R^2(\sin 2A + \sin 2B + \sin 2C)$,

and $(a+b+c) = 2R(\sin A + \sin B + \sin C)$;

$$\therefore r = \frac{R}{2}\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = \frac{R}{2}.\frac{\sin A \sin B \sin C}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= 4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}.$$

3. Let $CD = p$, fig. (1), $AD = x$, $BD = y$;

then $AC = \sqrt{p^2 + x^2}$, $BC = \sqrt{p^2 + y^2}$,

and $AC + BC = \sqrt{p^2 + x^2} + \sqrt{p^2 + y^2} = a$ suppose...(1),

and $AD - DB = x - y = \delta$ suppose...(2),

solving (1) and (2), we find by substituting in the expressions for AC and BC,

$$AC = \sqrt{p^2 + \tfrac{1}{4}\left(\delta \pm a\sqrt{1 - \frac{4p^2}{a^2 - \delta^2}}\right)^2},$$

$$BC = \sqrt{p^2 + \tfrac{1}{4}\left(-\delta \pm a\sqrt{1 - \frac{4p^2}{a^2 - \delta^2}}\right)^2}.$$

4. Let $t = \tan A$, $t' = \tan B$, fig. (1), $CD = p$,

then $t \cdot t' = \dfrac{p}{AD} \cdot \dfrac{p}{BD} = \dfrac{p^2}{a^2}$, if $a^2 = AD.BD$;

$$\therefore \frac{t + t'}{1 - t.t'} = -\tan C = -m, \text{ if } m = \tan C,$$

and therefore $t + t' = m \cdot \left(\dfrac{p^2}{a^2} - 1\right)$.

Hence $$t = \tfrac{1}{2}\left\{m\left(\frac{p^2}{a^2} - 1\right) \pm \sqrt{m^2\left(\frac{p^2}{a^2} - 1\right)^2 - 4\frac{p^2}{a^2}}\right\}$$

$$t' = \tfrac{1}{2}\left\{m \cdot \left(\frac{p^2}{a^2} - 1\right) \mp \sqrt{m^2 \cdot \left(\frac{p^2}{a^2} - 1\right)^2 - 4\frac{p^2}{a^2}}\right\},$$

which give A and B.

5. Let r_1, r_2, r_3, be the radii of the circles, which respectively touch the sides a, b, c; then, putting

$$S = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)} \text{ where } 2s = a + b + c,$$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s - a}{S} + \frac{s - b}{S} + \frac{s - c}{S},$$

$$= \frac{3s - 2s}{S} = \frac{s}{S} = \frac{1}{r}.$$

6. If a, b be the sides to which α, β are drawn,

$$a^2 = \frac{a^2}{4} + b^2,\ \beta^2 = a^2 + \frac{b^2}{4};$$

$$\therefore\ \beta^2 - \frac{a^2}{4} = \frac{15}{16}a^2, \text{ and } \therefore\ a = 2\sqrt{\frac{4\beta^2 - a^2}{15}}.$$

$$\text{So } b = 2\sqrt{\frac{4a^2 - \beta^2}{15}};$$

$$\therefore\ \tan A = \frac{a}{b} = \sqrt{\frac{4\beta^2 - a^2}{4a^2 - \beta^2}}, \text{ and } \tan B = \sqrt{\frac{4a^2 - \beta^2}{4\beta^2 - a^2}}.$$

7. Let k = area, $2s$ = perimeter, A = the given angle;

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b).(s-c)}{s.(s-a)}};\qquad \therefore\ \frac{k}{\tan\frac{A}{2}} = s.(s-a);$$

$$\therefore\ a = s - \frac{k}{s.\tan\frac{A}{2}}, \text{ the side } \textit{opposite} \text{ to } \angle A.$$

$$\text{Again,}\quad k = \tfrac{1}{2}bc\sin A, \text{ or } bc = \frac{2k}{\sin A},$$

$$\text{and } b + c = 2s - a;$$

$$\therefore\ b - c = \pm\sqrt{(2s-a)^2 - \frac{8k}{\sin A}};$$

$$\therefore\ b = \tfrac{1}{2}\left\{(2s-a) \pm \sqrt{(2s-a)^2 - 8k\ \text{cosec}\ A}\right\},$$

$$c = \tfrac{1}{2}\left\{(2s-a) \mp \sqrt{(2s-a)^2 - 8k\ \text{cosec}\ A}\right\}.$$

8. Fig. (2), $\sin ACB = \frac{AB}{AC}\sin ABC$;

$$\therefore\ L\sin ACB = \log 7639 + L\sin 43^\circ 8' - \log 5780,$$
$$= 3.8830365 + 9.8348646 - 3.7619278,$$
$$= 9.9559733 = L\sin 64^\circ 38' \text{ nearly};$$

$\therefore\ ACB = 64^0\,38'$; and $\therefore\ BAC = 180^0 - 107^0\,46' = 72^0\,14'$.

And $BC = AB.\dfrac{\sin BAC}{\sin ACB}$;

$\therefore \log BC = \log 7639 + L\sin 72^0\,14' - L\sin 64^0\,38'$,

$= 3.8830365 + 9.9787770 + (.0440311 - 10)$,

$= 3.9058446 = \log 8050.9$;

$\therefore\ BC = 8050.9$ feet.

9. The angles being as 3, 4, 5, and their sum 180^0, the angles are 45^0, 60^0, 75^0.

And area of the triangle

$$= \frac{r}{2}(\text{sum of the sides})$$

$$= r^2\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)$$

$$= r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \quad \text{(Ex. 1, p. 93)}$$

$$= r^2\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C - 1}$$

$$= r^2\frac{\dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{1}{\sqrt{2}} + \dfrac{1}{2} + \dfrac{\sqrt{3}-1}{2\sqrt{2}} - 1}$$

$$= r^2\frac{3 + \sqrt{6} + \sqrt{3}}{1 + \sqrt{2} + \sqrt{3} - 2\sqrt{2}}$$

$$= r^2\sqrt{3}.\frac{1 + \sqrt{2} + \sqrt{3}}{1 - \sqrt{2} + \sqrt{3}}.$$

10. Let $CD = d$, $CE = x$. Fig. (3).

Then $x\,d\sin C = \dfrac{1}{3}ab\sin C$, or $x = \dfrac{ab}{3d}$;

$$\therefore\ DE^2 = d^2 + x^2 - 2dx\cos C = d^2 + \frac{a^2b^2}{9d^2} - \frac{a^2+b^2-c^2}{3},$$

$$\text{or}\quad DE = \frac{1}{3d}\sqrt{a^2b^2 + 3d^2(c^2 + 3d^2 - a^2 - b^2)}.$$

11. Let A, B be the given angles and a the given side. Fig. (1).

$$\text{Area} = \tfrac{1}{2}CD \times AB = \tfrac{1}{2}a\sin B \times a\frac{\sin C}{\sin A}$$

$$= \frac{a^2}{2}\frac{\sin B}{\sin A}\sin(A+B).$$

12. If D be the point, and α, β, γ its distances from A, B, C, Fig. (4)

$$c^2 = a^2 + \beta^2 - 2a\beta\cos 120^0 = a^2 + a\beta + \beta^2;$$

$$\therefore\ c = \sqrt{a^2 + a\beta + \beta^2},$$

$$\text{so } b = \sqrt{a^2 + a\gamma + \gamma^2},\ \text{and}\ a = \sqrt{\beta^2 + \beta\gamma + \gamma^2}.$$

And the area ABC

= the sum of the areas BDC, CDA, ADB

$$= \tfrac{1}{2}(a\beta + a\gamma + \beta\gamma)\sin 120^0$$

$$= \frac{\sqrt{3}}{4}\cdot\{a\beta + a\gamma + \beta\gamma\}.$$

13. Let $x - y$, x, $x + y$ be the sides,

R, r the radii of the circumscribed and inscribed circles;

then, putting $x - y$, x and $x + y$ for a, b, c in the expressions for R and r (Arts. 47, 48), we have

$$R = \frac{x^2 - y^2}{\sqrt{3(x^2 - 4y^2)}},\quad r = \sqrt{\frac{x^2 - 4y^2}{12}};$$

$$\therefore\ x^2 - y^2 = (\text{distance between the centres})^2$$

$$= R^2 - 2Rr \quad \text{(Ex. 25, p. 115)}$$

$$= \frac{(x^2 - y^2)^2}{3(x^2 - 4y^2)} - \tfrac{1}{3}(x^2 - y^2).$$

Hence, $3x^2 = 15y^2$ or $x^2 = 5y^2$, and therefore the sides are

$$y(\sqrt{5} - 1),\ y\sqrt{5},\ y(\sqrt{5} + 1);$$

i. e. they are as $\sqrt{5} - 1,\ \sqrt{5},\ \sqrt{5} + 1$.

14. Let $x = AD$, $y = BD$, Fig. (5)

$$\frac{x}{y} = \frac{AC}{BC} = \tan\frac{B + B}{2} = \tan\frac{90 + B - A}{2} = \frac{1 + t}{1 - t}.$$

15. Let r = rad. of the circle inscribed in the triangle ABC; then, $AB = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)$, and so for BC and AC.

16. The angles being as 2, 3, 4 and their sum = 180^0, they are 40^0, 60^0, 80^0;

$$\therefore\ \frac{a + c}{b} = \frac{\sin 40^0}{\sin 60^0} + \frac{\sin 80^0}{\sin 60^0} = 2\cos\tfrac{1}{2}40^0 = 2\cos\frac{A}{2}.$$

17. The angles being as 1, 2, 3 and their sum = 180^0, they are 30^0, 60^0, 90^0; and the triangle therefore is right-angled. Hence ABC being the triangle right-angled at C,

$$\text{its area} = \tfrac{1}{2}AC \,.\, BC = \frac{p^2}{2}\operatorname{cosec} 30^0 .\ \operatorname{cosec} 60^0 = \frac{2p^2}{3}\sqrt{3}.$$

18. If $n - 1$, n, $n + 1$ be the sides, and A the least angle,

$$\frac{n + 1}{n - 1} = \frac{\sin 2A}{\sin A} = 2\cos A = \frac{(n + 1)^2 + n^2 - (n - 1)^2}{n\,.\,(n + 1)} = \frac{n + 4}{n + 1};$$

$$\therefore (n+1)^2 = (n-1).(n+4) \text{ or } n = 5;$$

therefore the sides are 4, 5, 6;

the angles are $82^0.50'$, $55^0.45'$ and $41^0.25'$.

19. The point (O) is the centre of the circumscribed circle, let ABC be the triangle,

then p, q, r being the perpendiculars upon the sides a, b, c,

$$\frac{\frac{1}{2}a}{p} = \tan \tfrac{1}{2} BOC = \tan A;$$

$$\therefore \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 2(\tan A + \tan B + \tan C)$$

$$= 2 \tan A \tan B \tan C$$

$$= 2 \frac{a}{2p} \frac{b}{2q} \frac{c}{2r}$$

$$= \frac{abc}{4pqr}.$$

20. The sides were found as well as the angles in Prob. 6.

21. The perpendicular from the angle C on AB being p, and D the diameter,

$$p = a . \sin B, \text{ and } D \sin B = b; \therefore Dp = ab.$$

22. O being the centre of the inscribed circle and r = its radius, Fig. (6),

$$r = DC = CE; \text{ also } AB = AE + BD,$$

i.e. $AC + BC$ exceeds AB by $EC + CD$, i.e. by $2r$.

23. If $ABC\ A'B'C'$ be the triangles,

$$A = A', \text{ and } B + B' = 180^0,$$

$$\text{then } \sin A = \sin A', \ \sin B = \sin B';$$

$$\therefore \frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin A'}{\sin B'} = \frac{a'}{b'}.$$

24. If a, b be the given sides, $A - B$ is the given difference;

$$\therefore \tan\frac{A+B}{2} = \frac{a+b}{a-b}\tan\frac{A-B}{2}, \text{ is known,}$$

$$\text{and } A = \frac{A+B}{2} + \frac{A-B}{2}, \quad B = \frac{A+B}{2} - \frac{A-B}{2}$$

are therefore known.

$$\text{Also } C = 180^0 - (A+B), \text{ and } c = a.\frac{\sin C}{\sin A}.$$

25. If A be the given angle, K = the area, and $2S$ the perimeter,

$$r = \text{rad. of inscribed circle} = \frac{K}{S},$$

$$2S - a = b + c = a + 2r\cot\frac{A}{2};$$

$$\therefore a = \tfrac{1}{2}\left\{2S - 2r\cot\frac{A}{2}\right\} = S - \frac{K}{S}\cot\frac{A}{2};$$

$$\therefore b + c = S + \frac{K}{S}\cot\frac{A}{2}, \quad \text{also } bc = \frac{2K}{\sin A};$$

$$\therefore b - c = \sqrt{\left(S + \frac{K}{S}\cot\frac{A}{2}\right)^2 - \frac{8K}{\sin A}} = M \text{ suppose;}$$

$$\therefore b = \tfrac{1}{2}\left(S + \frac{K}{S}\cot\frac{A}{2} + M\right)$$

$$c = \tfrac{1}{2}\left(S + \frac{K}{S}\cot\frac{A}{2} - M\right).$$

$$\text{Also } \sin B = \frac{b}{a}\sin A, \text{ and } C = 180^0 - A - B.$$

26. Let $ACB = \gamma$ the given angle; a, b the given sides, CD being drawn to the middle point of AB.

Let $\angle BCD = \theta$, $CDB = \phi$,

$$\frac{\sin\theta}{\sin\phi} = \frac{BD}{BC}, \quad \frac{\sin(\gamma-\theta)}{\sin(180^0-\phi)} = \frac{AD}{AC};$$

$$\therefore \frac{\sin(\gamma-\theta)}{\sin\theta} = \frac{a}{b}, \text{ or } \sin\gamma\cot\theta - \cos\gamma = \frac{a}{b};$$

$$\therefore \cot\theta\tan\gamma = 1 + \frac{a}{b}\sec\gamma = 1 + \tan^2\psi \text{ suppose};$$

$$\therefore \cot\theta = \cot\gamma\sec^2\psi, \text{ where } \psi \text{ is given by}$$

$$\tan^2\psi = \frac{a}{b}\sec\gamma.$$

27. If x, y, z be the angles,

$$x - z = \alpha \text{ the given difference,}$$

$$\text{then } z = x - \alpha, \; y = 180^0 - (2x - \alpha),$$

and, by Ex. 20, p. (89), $\cos\alpha \cdot \cos y = \cos(x + z - y)$,

$$\text{or } \cos\alpha \cdot \cos y = \cos(2x - \alpha - y) = \cos(180^0 - 2y),$$

$$\text{or } 2\cos^2 y + \cos\alpha\cos y - 1 = 0;$$

$$\therefore \cos y = \tfrac{1}{4}(-\cos\alpha \pm \sqrt{8 + \cos^2\alpha}) \text{ which gives } y:$$

$$\text{also } x = 90^0 + \frac{\alpha - y}{2}, \quad z = 90^0 - \frac{\alpha + y}{2}.$$

$$\text{Then } \frac{a}{\sin x} = \frac{b}{\sin y} = \frac{c}{\sin z}$$

give the remaining two sides when any one is given.

28. $a + b + c = 2R(\sin A + \sin B + \sin C)$

$$= \frac{r}{2} \cdot \frac{\sin A + \sin B + \sin C}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \text{ by Ex. 2, p. 93.}$$

$$\text{Or } \frac{a+b+c}{2r} = \frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}.$$

29. If C be the given angle, c the base, and $a - b = \delta$ the given difference;

$$\text{since } \frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}};$$

$$\therefore \sin\frac{A-B}{2} = \frac{\delta}{c}\cos\frac{C}{2}, \text{ and } \frac{A+B}{2} = 90^0 - \frac{C}{2},$$

$$\text{and } A = \frac{A+B}{2} + \frac{A-B}{2}, \; B = \frac{A+B}{2} - \frac{A-B}{2}.$$

30. If $x - y$, x, $x + y$ be the sides of the triangle,

$$\text{its area} = \sqrt{\frac{3x}{2}\left(\frac{x}{2}+y\right)\frac{x}{2}\left(\frac{x}{2}-y\right)};$$

also each side of the equilateral triangle of the same perimeter $= x$,

$$\text{and its area } \therefore = \sqrt{\frac{3x}{2}\left(\frac{x}{2}\right)^3};$$

$$\therefore \frac{8}{5} = \sqrt{\frac{x^2-4y^2}{x^2}}, \text{ or } 16x^2 = 100y^2, \text{ or } \frac{x}{y} = \frac{5}{2};$$

$$\therefore x - y : x : x + y = \frac{3}{2}y : \frac{5}{2}y : \frac{7}{2}y = 3 : 5 : 7,$$

and if θ be angle opposite $x + y$, i. e. the greatest angle,

$$\cos\theta = \frac{5^2 + 3^2 - 7^2}{2 \times 5 \times 3} = -\frac{15}{30} = -\frac{1}{2} = \cos 120^0;$$

$$\therefore \theta = 120^0.$$

31. See HYMERS' TRIGONOMETRY, p. 139.

32. If x, $\frac{x}{2}$, $\frac{x}{4}$ be the angles, $x = \frac{4}{7} \cdot 180^0$,

and if a, b, c be the sides, a the greatest, and $2s = a + b + c$,

$$\tan\frac{B}{2}\tan\frac{C}{2}, \text{ or } \tan\frac{x}{4}\tan\frac{x}{8}$$

$$= \sqrt{\frac{(s-a)(s-c)}{s(s-b)} \cdot \frac{(s-a)(s-b)}{s(s-c)}} = \frac{s-a}{s};$$

$$\therefore\ a = s \cdot \left(1 + \tan\frac{180^0}{7}\tan\frac{90^0}{7}\right).$$

33. $U = \sin nA + \sin nB + \sin nC$

$$= 2\sin n\frac{A+B}{2}\cos n\frac{A-B}{2} + \sin nC,$$

or if n be of one of the forms $4m + 1$, $4m + 3$

$$= \pm 2\cos\frac{nC}{2}\left(\cos n\frac{A-B}{2} + \cos n\frac{A+B}{2}\right)$$

$$= \pm 4\cos\frac{nC}{2}\cos\frac{nB}{2}\cos\frac{nA}{2}.$$

But if n be of one of the forms $4m$, $4m + 2$,

$$U = \pm 2\sin\frac{nC}{2} \cdot \left(\cos n\frac{A-B}{2} - \cos n\frac{A+B}{2}\right)$$

$$= \pm 4\sin\frac{nC}{2}\sin\frac{nB}{2}\sin\frac{nA}{2}.$$

34. The triangle in this question is impossible.

HEIGHTS AND DISTANCES.

1. $BC = 200 \cdot \tan 25^0.10'$;

$\therefore \log BC = \log 200 + L \tan 25^0.10' - 10 = \log 93.97$ nearly;

$\therefore BC = 94$ feet nearly.

2. Let $x = DE$ be the height of the balloon, Fig. (7),

α, β, γ its altitudes at A, B, C and $AB = BC = a$.

Adding the squares of DA and DC obtained by means of Art. 30 from the triangles DBA, DBC, we have

$$AD^2 + DC^2 = 2DB^2 + 2AB^2, \quad \therefore$$

$$\text{or } x = a\sqrt{\frac{2}{\cot^2\alpha + \cot^2\gamma - 2\cot^2\beta}}.$$

3. If $x = AB$ the height of the mountain, Fig. (8),

CD the base, $\angle BCD = 58^0.29'$, $CDB = 111^0.52'$;

$$\therefore CBD = 9^0.39',$$

$$\text{and } BDA = 11^0.18',$$

$$x = CD\frac{\sin BCD}{\sin CBD} \cdot \sin BDA,$$

$$\log x = \log 2761 + L \sin 58^0.29' + L \sin 11^0.18' - L \sin 9^0.39' - 10$$

$$= 3.4395419 = \log 2751.325,$$

$$x = 2751\tfrac{1}{3} \text{ feet nearly.}$$

4. If AB be the tower, Fig. (9),

$$\angle ACB = 55^0, \; ADB = 48^0,$$

and therefore $DAC = 7^0$, and $DC = 30$ feet,

$$BC = AC \cdot \cos ACB = DC\frac{\sin ADC}{\sin DAC} \cdot \cos ACB;$$

$$\therefore \log BC = \log 30 + L\sin 48^0 - L\sin 7^0 + L\cos 55^0 - 10$$

$$= \log 104.928,$$

$$BC = 104.928 \text{ feet.}$$

5. Let $AB = x$ (fig. 10) be the tower, $CD = a$ the pillar, E the place of observation on the horizontal plane, $\alpha = \angle AEC$.

At A let the angle $CAE = \beta$ be observed, and γ, δ the angles of depression of C, E; then

$$AC = (x - a)\operatorname{cosec}\gamma, \quad AE = x\operatorname{cosec}\delta;$$

$$\therefore \frac{\sin\alpha}{\sin(\alpha+\beta)} = \frac{AC}{AE} = \frac{x-a}{x}\cdot\frac{\sin\delta}{\sin\gamma};$$

$$\therefore x = a\cdot\frac{\sin\delta\sin(\alpha+\beta)}{\sin\delta\sin(\alpha+\beta) - \sin\alpha\sin\gamma}.$$

6. Let AB (fig. 9) be the tower; 45^0, 30^0 the depressions of C and D, then $CB = AB = 50$ feet.

$$DB = 50\tan 60^0 = 50\sqrt{3},$$

$$\therefore DC = DB - BC = 50(\sqrt{3} - 1).$$

$$AC = 50\sqrt{2},$$

$$AD = 50\operatorname{cosec} 30^0 = 100 \text{ feet.}$$

7. From (fig. 11) $DB = EB$, and therefore the perpendicular BF on AC bisects DE;

$$\therefore AF = a + \frac{b}{2}.$$

Also the angles are 30^0, 30^0, 120^0;

$$\therefore BC = AB = AF\operatorname{cosec} 60^0$$

$$= \frac{2\sqrt{3}}{3}\left(a + \frac{b}{2}\right) = (2a + b)\frac{\sqrt{3}}{3}.$$

8. If EC (fig. 12) $= a$, $\alpha = \angle ACF$, $\beta = BCG$, $\gamma = ACE$, $\delta = ACB$, $\epsilon = AEC$, then

$$AF = AC \cdot \sin\alpha = a \cdot \frac{\sin\epsilon}{\sin(\gamma + \epsilon)} \cdot \sin\alpha.$$

$$\text{So } BG = a \cdot \frac{\sin\epsilon}{\sin(\gamma - \delta + \epsilon)} \cdot \sin\beta.$$

9. This is the same as (6).

10. If AB (fig. 13) be the tower, $AC = 80$ feet measured along the declivity; $ACB = 30^0$, and $ACD = 15^0$, if CD be horizontal, then

$$AB = 2AC\sin 15^0 = \frac{\sqrt{3}-1}{\sqrt{2}} \times 80 = 40\sqrt{2}(\sqrt{3}-1) = 41 \text{ feet.}$$

11. Fig. (14). If $AB = a$, $BC = b$, and θ, $90^0 - \theta$, 2θ be the elevations of the steeple at A, B, C,

$$DE = (a + b)\sin 2\theta;$$

and since angle $DEB = 90^0 - DBE = \theta$,

$$DE\tan\theta + a = DE\cot\theta;$$

$$\text{or } \{(a+b)\sin 2\theta\} \cdot \tan\theta + a = \{(a+b)\sin 2\theta\} \cdot \cot\theta,$$

$$2(a+b)\sin^2\theta + a = 2(a+b)\cos^2\theta,$$

$$(a+b)\cos 2\theta = \frac{a}{2};$$

$$\therefore DE = (a+b)\sin 2\theta = \sqrt{(a+b)^2 - \frac{a^2}{4}}.$$

12. Fig. (15). Let $AB = 108$, and $CD = x$ be the heights of the tower and column,

$\angle EAC = 30^0$, $\angle EAD = 60^0$, and $\therefore \angle CAD = CDA$.

Hence $CA = CD$; also $\angle CAB = 60^0$,

$$\therefore (108 - x) \cdot \sec 60^0 = x,$$

$$\text{or } 3x = 216, \; x = 72 \text{ feet.}$$

13. Fig. (16). If $a = AB$, $\angle DAB = \alpha_1$, $DAC = \alpha_2$, $DBA = \beta_1$, $DBC = \beta_2$, $DCA = \gamma_1$, $DCB = \gamma_2$,

$$DA = AB \cdot \frac{\sin DBA}{\sin ADB} = a \cdot \frac{\sin \beta_1}{\sin (\alpha_1 + \beta_1)};$$

$$\therefore AC = \left\{ a \cdot \frac{\sin \beta_1}{\sin (\alpha_1 + \beta_1)} \right\} \cdot \frac{\sin (\alpha_2 + \gamma_1)}{\sin \gamma_1}.$$

So $$BC = a \cdot \frac{\sin \alpha_1}{\sin (\alpha_1 + \beta_1)} \cdot \frac{\sin (\beta_2 + \gamma_2)}{\sin \gamma_2},$$

$$DC = a \cdot \frac{\sin \beta_1}{\sin (\alpha_1 + \beta_1)} \times \frac{\sin \alpha_2}{\sin \gamma_1},$$

and therefore if $DE = h$, $CE = \sqrt{DC^2 - h^2}$ is known, which is reduced to a form fit for logarithms by putting $\frac{h}{DC} = \sin \phi$.

14. Fig. (17). If AB be the tower, x its height, ACB, ACD, the observed angles $= \alpha$, β, and $BD = a$ the distance from the road CDE; then since ADC is a right angle,

$$AD = AC \sin \beta = \frac{x \sin \beta}{\sin \alpha};$$

$$\therefore a^2 + x^2 = AD^2 = x^2 \frac{\sin^2 \beta}{\sin^2 \alpha},$$

$$x = \frac{a \cdot \sin \alpha}{\sqrt{\sin^2 \beta - \sin^2 \alpha}} = \frac{a \cdot \sin \alpha}{\sqrt{\sin (\beta + \alpha) \cdot \sin (\beta - \alpha)}}.$$

15. Fig. (18). Measure the distance $AB = a$ between two objects on the accessible bank, and from A, B observe the angular distances α, β of B and A from any well-defined object C on the other bank, and let CD be perpendicular to AB, then the breadth $CD = CB \sin \beta = a \frac{\sin \alpha \cdot \sin \beta}{\sin (\alpha + \beta)}$.

16. Fig. (19). Let the plane of the paper be the vertical plane through A, B, C; BE horizontal, AE vertical.

$$AE = AB \sin 60^0 = \frac{\sqrt{3}}{2} BC \frac{\sin 135^0}{\sin 15^0} = \frac{\sqrt{3}}{2} \frac{2}{\sqrt{3}-1} \text{ miles}$$

$$= (3 + \sqrt{3})\ 880 \text{ yards} = 4164.204 \text{ yards.}$$

17. Fig. (20). If AB be the object, BC the tower, D the observer,

$$.015 = \frac{\tan ADC - \tan BDC}{1 + \tan ADC \tan BDC} = \frac{6 \times 100}{(100)^2 + x(x+6)},$$

$$x^2 + 6x = 30000, \quad x = 170.23 = 170\tfrac{1}{4} \text{ ft. nearly.}$$

ILLUSTRATIONS OF THE CIRCULAR MEASURE.

Obs. The ratio of the numerical representatives of two angles will be the same, whatever be the unit to which the angles are referred. Hence, since the circular measure of an angle containing 180^0 is π, (3.14159), we can always by a simple proportion transfer from one system to another.

1. Let x be the number of degrees in an angle whose circular measure is $\frac{\pi}{7}$, then

$$\pi : \frac{\pi}{7} :: 180 : x, \text{ whence } x = 25^0.\ 42'\ \tfrac{6}{7}.$$

The equivalents to the remaining quantities are $85^0.\ 56'.\ 37''$, $114^0.\ 35'.\ 29''$, $237^0.\ 17'.\ 45''$, and $179^0.\ 54'.\ 32''$.

2. Ans. .244346, .270506, 2.09438, .3, 1.6065.

3. $\frac{2}{3} \times 90 : 45 :: 1 : x$ (the required value), whence

$$x = \tfrac{3}{4}.$$

4. Since $1 = 2 \dfrac{\text{arc subtending the } \textit{unit} \text{ of angular measure}}{\text{radius}}$,

this arc $= \frac{1}{2}$ radius:

or the unit is the angle subtended at the centre of the circle by an arc equal to half the radius.

5. Ans. .00029.

6. Let n be the number of seconds in the angle whose circular measure is θ, then

$$\frac{n}{1} = \frac{\theta}{\text{circular measure of } 1''} = \frac{\theta}{\sin 1''}; \quad \therefore \theta = n \sin 1'',$$

$$\text{and} \quad \therefore \sin\theta = n \sin 1'' - \tfrac{1}{6}(n \sin 1'')^3.$$

MISCELLANEOUS PROBLEMS.

1. If $a = PA$, $b = PB$, $c = PC$, $x = AB$, $\theta = \angle ABP$, P being the point within the square $ABCD$, then,

$$a^2 = b^2 + x^2 - 2bx\cos\theta, \text{ from } \Delta ABP,$$

$$c^2 = b^2 + x^2 - 2bx\sin\theta, \text{ from } \Delta PBC.$$

Eliminate θ, and solve the resulting equation, then

$$x^2 = \tfrac{1}{2}\{a^2 + c^2 \pm \sqrt{4b^2(a^2 + c^2 - b^2) - (a^2 - c^2)^2}\}.$$

2. If $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = x$, Fig. (21)

$$\frac{a^2 + b^2 - x^2}{2ab} = \cos ABC = -\cos ADC = \frac{x^2 - c^2 - d^2}{2cd},$$

$$x^2 = \frac{(a^2 + b^2).cd + (c^2 + d^2).ab}{ab + cd},$$

$$x = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}}.$$

3. $\sin\theta - \text{vers}\,\theta = \sin\theta + \cos\theta - 1 = \sqrt{1 + \sin 2\theta} - 1,$

which is positive while θ lies between 0 and 90^0, and is greatest when $2\theta = 90^0$, or $\theta = 45^0$; and then the excess $= \sqrt{2} - 1$.

4. Let $ABCD$ be the figure, $AB = a$, $BC = b$, $CD = c$, $DA = d$.

Let (b, a) denote the $\angle$ between the sides b, a, &c....

$$a = d\cos(d, a) + c\cos(c, a) + b\cos(b, a);$$

$$\therefore\ 2a^2 = 2ad\cos(d, a) + 2ac\cos(c, a) + 2ab\cos(b, a).$$

Also $\quad a^2 + d^2 - 2ad\cos(a, d) = b^2 + c^2 - 2bc\cos(b, c);$

$$\therefore\ d^2 = a^2 + b^2 + c^2 - 2\{ac\cos(a, c) + ab\cos(a, b) + bc\cos(b, c)\}.$$

5. Fig. (22). If $ABCD$ be the base, $AB = a$, E the vertex, EF perpendicular to the base meeting it in F, the intersection of the diagonals; then if K be the bisection of AB, $\angle EKF$ is the inclination required $= I$ suppose;

$$\therefore\ \cos I = \frac{FK}{EK} = \frac{\frac{1}{2}a}{\frac{a\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

6. Fig. (23). If A, B, C, a, b, c be the corresponding positions of the two ships, AN the north line through A, $\theta = NAC$,

$\frac{ab}{bc} = \frac{p}{q} = \frac{AB}{BC}$; $\therefore$ Aa, Bb, Cc will meet in a point (O).

$$\frac{p}{q} = \frac{AB}{BO}\cdot\frac{BO}{BC} = \frac{\sin\{(\theta-\beta)-(\theta-\alpha)\}}{\sin(\theta-\alpha)}\cdot\frac{\sin(\theta-\gamma)}{\sin\{(\theta-\gamma)-(\theta-\beta)\}},$$

$$\text{or if } \frac{p}{q}\cdot\frac{\sin(\beta-\gamma)}{\sin(\alpha-\beta)} = \lambda,\ \lambda\sin(\theta-\alpha) = \sin(\theta-\gamma);$$

$$\text{and } \therefore\ \tan\theta = \frac{\lambda\sin\alpha - \sin\gamma}{\lambda\cos\alpha - \cos\gamma}.$$

7. Fig. (24). The sides of $\Delta O_1 O_2 O_3$ are $\sqrt{3}$, 2, 1; and the area

$$= \sqrt{\frac{3+\sqrt{3}}{2} \cdot \frac{3-\sqrt{3}}{2} \cdot \frac{\sqrt{3}+1}{2} \cdot \frac{\sqrt{3}-1}{2}} = \frac{\sqrt{3}}{2};$$

and if $r_1 = \frac{1}{2}(\sqrt{3}-1)$, $r_2 = \frac{1}{2}(\sqrt{3}+1)$, $r_3 = \frac{1}{2}(3-\sqrt{3})$,

$$\Delta O_1 AB = \frac{r_1^2}{(r_1+r_2)(r_1+r_3)} \Delta O_1 O_2 O_3 = \frac{1}{4} \frac{4-2\sqrt{3}}{\sqrt{3}\times 1} \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{4}.$$

So $\Delta O_2 BC = \frac{1}{8}(2+\sqrt{3})$, and $\Delta O_3 CA = \frac{1}{8}(6\sqrt{3}-9)$;

$$\therefore \Delta ABC = \frac{\sqrt{3}}{2} - (\tfrac{5}{8}\sqrt{3} - \tfrac{3}{8}) = \frac{3-\sqrt{3}}{8}.$$

8. If A_1, A_2 be the areas of the outer and inner polygons of n sides,

$$A_1 : A_2 :: \tan\frac{\pi}{n} : \sin\frac{\pi}{n}\cos\frac{\pi}{n} :: 1 : \cos^2\frac{\pi}{n};$$

$$\therefore \tfrac{1}{3} \text{ or } \tan^2\frac{\pi}{6} = \frac{A_1 - A_2}{A_2} = \tan^2\frac{\pi}{n}; \quad \therefore n = 6.$$

9. Fig. (25). Let A, B be the points of observation, C, D the headlands, AE, CE' lines drawn eastward.

$$BD = AB = 12 \text{ miles}; \quad \therefore CB = 12 \operatorname{cosec}\frac{45^0}{2} = 12\sqrt{\frac{2\sqrt{2}}{\sqrt{2}-1}};$$

$$\therefore CD = \sqrt{CB^2 + BD^2} = 12\sqrt{\frac{3\sqrt{2}-1}{\sqrt{2}-1}} = 12\sqrt{7-4\sqrt{2}},$$

$$\text{and } \angle E'CD = 45^0 - \sin^{-1}\frac{BD}{CD} = 45^0 - \operatorname{cosec}^{-1}\sqrt{7-4\sqrt{2}}.$$

10. Retaining the notation of (8), and putting $n = 10$,

$$A_1 : A_2 = 1 : \cos^2\frac{\pi}{10} = 1 : \cos^2 18^0 = 8 : 5+\sqrt{5}.$$

11. The cosecants being in harmonical progression, the sines are in arithmetical progression;

$\therefore \sin A - \sin B = \sin B - \sin C$, and $A + B + C = 180^0$;

$$\therefore \cos\frac{A+B}{2}\sin\frac{A-B}{2} = \cos\frac{B+C}{2}\sin\frac{B-C}{2},$$

$$\text{or } \sin\frac{C}{2}\sin\frac{A-B}{2} = \sin\frac{A}{2}\sin\frac{B-C}{2},$$

$$\sin\frac{A}{2} : \sin\frac{C}{2} = \sin\frac{A-B}{2} : \sin\frac{B-C}{2};$$

12. If a = each side, r = radius of the circle, $t = \tan\theta$,

$$\tfrac{1}{2}\frac{a}{r} = \sin 2\theta = 2\sin\theta\cos\theta = \frac{2t}{1+t^2},$$

$$\therefore a : 2r = 2t : 1 + t^2.$$

13. Fig. (26). If DE be the perpendicular from the vertex D on the base ABC, and EF perpendicular to AB, then DF is perpendicular to AB; and by symmetry E is equidistant from the sides of ABC.

$\therefore EF \times (3a) = 2k$, (if k = area of base, and a = each side),

$$\text{also } k = \tfrac{1}{2}a^2\sin 60^0 = a^2\frac{\sqrt{3}}{4};$$

$$\therefore DE^2 = DF^2 - EF^2 = \left(a\frac{\sqrt{3}}{2}\right)^2 - \frac{4k^2}{9a^2} = \frac{8}{9}\sqrt{3}\,.\,k,$$

$$DE = \tfrac{2}{3}\,12^{\frac{1}{4}}\,.\,k^{\frac{1}{2}}.$$

14. Fig. (27). Since F is the bisection of AB, and E of AC, EF is parallel to BC and $= \frac{1}{2}BC$, so for DE and DF; and EDF is an equilateral triangle a side of which $= \frac{1}{2}$ side of ABC;

$\therefore$ the radius of the circle inscribed in $EDF = \frac{1}{2}$ (that in ABC)

Hence $r_1 = \frac{1}{2}r$, so $r_2 = \frac{1}{2}r_1$, $r_3 = \frac{1}{2}r_2$, &c....

$$\therefore r_1 + r_2 + r_3 + \ldots \textit{ad inf.} = r\left(\frac{1}{2} + \frac{1}{2^2} + \ldots \infty\right) = r.$$

15. See HYMERS' TRIGONOMETRY, p. (75).

16. If $2s$ = sum of the sides, $a = 3x$, $b = 5x$, $c = 6x$, $s = 7x$, and $\frac{r}{R} = 4\frac{(s-a)(s-b)(s-c)}{abc} = \frac{16}{45}$.

17. Since triangles on the same base are as their altitudes,

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = \frac{\triangle OBC}{ABC} + \frac{OCA}{ABC} + \frac{OAB}{ABC} = 1.$$

18. Fig. (28). If x = each side of the square; a, b the sides opposite to the angles A and B,

$$\frac{x}{a} = \frac{AF}{AB} = 1 - \frac{BF}{AB} = 1 - \frac{x}{b};$$

$\therefore \frac{1}{a} + \frac{1}{b} = \frac{2}{2x}$, or a, $2x$, b are in harmonical progression.

19. If A_n, A'_n be the areas of the polygons described in and about the same circle,

$$A_n = nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n}, \quad A'_n = nr^2 \tan\frac{\pi}{n},$$

$$\text{and } A_{\frac{n}{2}} \times A'_{\frac{n}{2}} = \left(\frac{n}{2}r^2\right)^2 \times \sin\frac{2\pi}{n}\cos\frac{2\pi}{n} \times \tan\frac{2\pi}{n} = A_n^2;$$

$\therefore A_n$ is a mean proportional between $A_{\frac{n}{2}}$ and $A'_{\frac{n}{2}}$.

$$\text{Also } \frac{1}{A_n} + \frac{1}{A'_{\frac{n}{2}}} = \frac{1}{nr^2} \cdot \left\{ \frac{1}{\sin\frac{\pi}{n}\cos\frac{\pi}{n}} + \frac{2\cos\frac{2\pi}{n}}{\sin\frac{2\pi}{n}} \right\}$$

$$= \frac{2}{nr^2}\cot\frac{\pi}{n} = \frac{2}{A'_n};$$

$\therefore A_n$, A'_n, $A'_{\frac{n}{2}}$ are in harmonical progression.

20. Each side $= 2r \sin \frac{180^0}{5} = 2r \sin 36^0 = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$,

$$r = \frac{2\sqrt{10 + 2\sqrt{5}}}{\sqrt{80}} = \sqrt{\frac{5 + \sqrt{5}}{10}}.$$

21. If O be the centre of the circumscribed circle, and R = radius, then $a = 2R \sin A$, and so for b and c.

22. If a, a' be sides of the polygons of n, $2n$ sides respectively, then $2na' = na$ or $a = 2a'$;

$$\therefore\ R \sin \frac{\pi}{n} = 2R' \sin \frac{\pi}{2n}, \text{ or } R \cos \frac{\pi}{2n} = R'; \ldots (1)$$

and since R', r' are the radii of the circles described about and in the same polygon of $2n$ sides; $\therefore\ r' = R' \cos \frac{\pi}{2n}$ ... (2)

By (1) × (2), $Rr' = R'^2$.

Again, $R + r = R + R \cos \frac{\pi}{n} = 2R \cos^2 \frac{\pi}{2n} = 2r'$.

23. Six equilateral triangles, since each angle is 60^0, will fill up space about a point. In a square each angle is 90^0, therefore four will fill plane space about a point. In a regular pentagon each angle is 108^0, and space about a point cannot be filled by any number of them.

In a regular hexagon each angle is 120^0; therefore three will fill the space about a point.

In a regular polygon of seven sides each $\angle = \frac{900^0}{7}$; therefore three such polygons placed with a common point will more than fill 360^0: and so for all of more sides.

Hence, an indefinite area, &c. Q. E. D.

24. For if not, let Aa, Bb, meet in O, join CO and produce it to meet AB in c', then

$$\frac{\triangle AOC}{\triangle BOC} = \frac{Ac'}{Bc'}, \quad \frac{\triangle BOC}{\triangle AOB} = \frac{Cb}{Ab}, \quad \frac{\triangle AOB}{\triangle AOC} = \frac{Ba}{Ca};$$

therefore, multiplying together, we have

$$\frac{Ac'}{Bc'} \cdot \frac{Cb}{Ab} \cdot \frac{Ba}{Ca} = 1,$$

and comparing this with the given equation, we have

$$\frac{Ac'}{Bc'} = \frac{Ac}{Bc}, \text{ and } \therefore \frac{Ac' + Bc'}{Bc'} = \frac{Ac + Bc}{Bc},$$

$$\text{or } \frac{AB}{Bc'} = \frac{AB}{Bc}; \quad \therefore Bc' = Bc,$$

or the point c must be on the line CO produced, i. e., Aa, Bb, Cc meet in a point.

25. Let O, O', (fig. 29) be the centres of the inscribed and circumscribed circles; draw ON, $O'N'$ perpendicular, and $O'n$ parallel to AB; then

$$OO'^2 = On^2 + O'n^2 = (r - R\cos C)^2 + (R\sin C - r\cot\tfrac{1}{2}A)^2,$$

$$= R^2 + r^2\operatorname{cosec}^2\tfrac{1}{2}A - 2Rr(\cos C + \sin C\cot\tfrac{1}{2}A),$$

or since $r^2 = 4Rr\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$, (Ex. 2, p. 93)

$$OO'^2 = R^2 - 2Rr\{(1 - 2\sin^2\tfrac{1}{2}C) + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C\cot\tfrac{1}{2}A$$

$$- 2\operatorname{cosec}\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\}.$$

$$= R^2 - 2Rr + 4Rr\operatorname{cosec}\tfrac{1}{2}A\sin\tfrac{1}{2}C\{\sin\tfrac{1}{2}B - \cos\tfrac{1}{2}(A + C)\}$$

$= R^2 - 2Rr$, since $\sin\frac{1}{2}B = \cos\frac{1}{2}(A + C)$.

26. If $x =$ this expression, and $\frac{b}{a} = \cos\theta$ (b being $< a$ when x is possible),

$$x = \frac{1}{\sqrt{a}}\left(\frac{\sqrt{1 - \cos\theta}}{1 + \cos\theta} + \frac{\sqrt{1 + \cos\theta}}{1 - \cos\theta}\right)$$

$$= \frac{1}{\sqrt{2a}}\frac{\sin\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta}(1 + \cot^2\tfrac{1}{2}\theta),$$

or if $\tan^2\phi = \cot^3\frac{1}{2}\theta$, since $\tan\phi$ may have any value,

$$x = \frac{1}{\sqrt{2a}}\tan\tfrac{1}{2}\theta\sec\tfrac{1}{2}\theta\sec^2\phi.$$

27. $n\{(1+x)-1\} = (\sqrt{1+x}-1)(\sqrt{1-x}+1)$,

$$n(\sqrt{1+x}+1) = \sqrt{1-x}+1.$$

Solving this equation

$$x = \frac{4n(1-n^2)}{(1+n^2)^2};$$

or putting $n = \tan\frac{1}{2}a$, $x = \sin 2a$.

28. Let the circle inscribed in $ABCD$ touch the sides AB, BC, CD, DA in E, F, G, H. Then, putting $AB = a$, $BC = b$, $CD = c$, $DA = d$,

$$d + b = (AE + DG) + (CG + BE) = c + a.$$

Also, if

$$2s = a + b + c + d = 2(a + c), \text{ or } 2(b + d),$$

$$\text{area in second case} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$$= \sqrt{abcd}.$$

29. This is the same as Ex. 7, p. (81).

30. Since $\tan\frac{2\pi}{3} = -\sqrt{3}$, $\tan\frac{4\pi}{3} = \sqrt{3}$;

$$\therefore \tan\tfrac{1}{3}(2\pi+\theta) + \tan\tfrac{1}{3}(4\pi+\theta) = \frac{8\tan\frac{1}{3}\theta}{1-3\tan^2\frac{1}{3}\theta};$$

$$\therefore \tan\tfrac{1}{3}\theta + \tan\tfrac{1}{3}(2\pi+\theta) + \tan\tfrac{1}{3}(4\pi+\theta)$$

$$= \tan\tfrac{1}{3}\theta \cdot \frac{9-3\tan^2\frac{1}{3}\theta}{1-3\tan^2\frac{1}{3}\theta}.$$

So $\cot\frac{1}{3}\theta + \cot\frac{1}{3}(2\pi+\theta) + \cot\frac{1}{3}(4\pi+\theta)$

$$= \cot\tfrac{1}{3}\theta \frac{9-3\cot^2\frac{1}{3}\theta}{1-3\cot^2\frac{1}{3}\theta},$$

and the product therefore

$$= 9 \cdot \frac{3-\tan^2\frac{1}{3}\theta}{1-3\tan^2\frac{1}{3}\theta} \frac{3\tan^2\frac{1}{3}\theta - 1}{\tan^2\frac{1}{3}\theta - 3} = 9.$$

31. Fig. (30). Let $AB = 32$ ft. be the pole, $CD = 6$ ft. the man, O the Earth's centre, and r its radius in miles; join BD, BD will touch the Earth's surface; and $BE + ED = AEC$ nearly; or, since $BE = \sqrt{BA.Ba}$, $ED = \sqrt{DC.Dc}$, and arc $AEC = 10$ ft.,

$$\sqrt{\frac{32}{3 \times 1760}\left(2r + \frac{32}{3 \times 1760}\right)} + \sqrt{\frac{6}{3 \times 1760}\left(2r + \frac{6}{3 \times 1760}\right)} = 10,$$

or nearly $\sqrt{r}(\sqrt{64} + \sqrt{12}) = 10\sqrt{3 \times 1760}$;

or $\sqrt{r} = 10\dfrac{\sqrt{3 \times 440}}{13}(4 - \sqrt{3})$,

$$r = \frac{100}{169} \times (3 \times 440) \times (19 - 8\sqrt{3})$$

$$= 4017.42 \text{ nearly.}$$

32. Fig. (30). If E be the observer's station, AB the ship's mast when first visible, then $EB = 12$ miles,

and nearly $12^2 = \dfrac{90}{3 \times 1760}\left(2r + \dfrac{90}{3 \times 1760}\right)$,

$$2r = 144 \times \frac{176}{3};$$

$$\therefore\ r = 24 \times 176 = 4224 \text{ miles.}$$

STATICS.

COMPOSITION AND RESOLUTION OF FORCES.

1. If P, Q, R be the forces

$$P : Q : R = \sin(Q, R) : \sin(P, R) : \sin(P, Q)^*$$

(Art. 11. Cor.)

$$= \sin 135^0 : \sin 120^0 : \sin 105^0$$

$$= 2 : \sqrt{6} : \sqrt{3} + 1.$$

2. The tack being smooth the forces P, Q are equal; and the direction of R, the pressure on the tack, bisects the angle θ between the directions of P and Q;

$$\therefore \frac{P}{R} = \frac{\sin(Q, R)}{\sin(P, Q)} = \frac{\sin \frac{1}{2}\theta}{\sin \theta} = \frac{\frac{1}{2}}{\cos \frac{1}{2}\theta}.$$

3. Fig. (31). The rod AB being without weight, the point B is kept at rest by 3 forces: the weight (W) of the body at B, the tension (T) of the string BC, and the action of the rod in direction of its length;

$$\therefore \frac{T}{W} = \frac{\sin WBD}{\sin CBD} = \frac{\cos CAB}{\sin CBA};$$

and the sides of the triangle ABC being known, these angles can be found, and T can be determined.

4. Fig. (32). If $\angle ACP = \theta$; and CA, PG be vertical; and PT tangent at P:

$$\frac{\text{force of gravity in } PT}{\ldots\ldots\ldots\ldots\ldots PA} = \frac{\cos GPT}{\cos GPA} = \frac{\sin \theta}{\sin \frac{1}{2}\theta} = 2\cos\frac{\theta}{2}.$$

* By (P, Q) is meant the angle between the directions of P and Q.

5. Let ACW (fig. 33) be the string fastened at A, $BC = b$ the rod, and $AB = a$; since the rod has no weight its action is wholly in the direction of its length, and therefore it bisects the angle ACW.

Let $\theta = \angle CBX$;

$$\therefore \angle BAC = \theta - \left(\frac{\pi}{2} - \theta\right) = 2\theta - \frac{\pi}{2},$$

$$\text{and } \frac{b}{a} = \frac{\sin\left(2\theta - \frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2} - \theta\right)} = -\frac{2\cos^2\theta - 1}{\cos\theta},$$

$$\cos^2\theta + \frac{b}{2a}\cos\theta = \tfrac{1}{2};$$

$$\therefore \cos\theta = \frac{1}{4a}(-b + \sqrt{8a^2 + b^2}).$$

6. If θ be the required angle, then the resultant of P and Q being $\frac{1}{2}(P + Q)$,

$$\tfrac{1}{4}(P + Q)^2 = P^2 + Q^2 + 2PQ\cos\theta,$$

$$\cos\theta = \frac{2PQ - 3P^2 - 3Q^2}{8PQ}.$$

7. Let θ be the angle, $3P$ and $2P$ the forces; and therefore $\sqrt{3P \times 2P}$ = resultant,

$$\cos\theta = \frac{P^2(3 \times 2) - 9P^2 - 4P^2}{2(3 \times 2)P^2} = -\tfrac{7}{12},$$

$$\theta = 125^0.42'.$$

8. Fig. (34) Let P represent the strength of each man; then, resolving the forces along and perpendicular to the tree, the horizontal forces are $4P \sin 60^0$ or $2P\sqrt{3}$ towards the South, and $6P \sin 45^0$ or $3P\sqrt{2}$ towards the South East. (Art. 13).

Or, resolving the latter, S. and E.,

the whole horizontal force S. $= (2\sqrt{3} + 3)\,P$,

and ..... E. $= 3P$.

Therefore if θ be the angle between the direction of the resultant force and the East,

$$\tan\theta = \tfrac{1}{3}(2\sqrt{3} + 3).$$

9. Let S be the sum of the forces P, Q; R their given resultant, α the angle between their directions: then

$$P^2 + Q^2 + 2PQ\cos\alpha = R^2, \text{ and } P + Q = S.$$

Which equations give

$$P = \tfrac{1}{2}(S + \sqrt{R^2\operatorname{cosec}^2\tfrac{1}{2}\alpha - S^2\cot^2\tfrac{1}{2}\alpha}),$$

$$Q = \tfrac{1}{2}(S - \sqrt{R^2\operatorname{cosec}^2\tfrac{1}{2}\alpha - S^2\cot^2\tfrac{1}{2}\alpha}).$$

10. If $P : Q : R :: \sqrt{3} + 1 : \sqrt{6} : 2$, then (Art. 10, Ex. 3)

$$\cos(P, Q) = \frac{R^2 - P^2 - Q^2}{2PQ} = -\frac{1}{\sqrt{2}} = \cos 135^0.$$

Therefore the angle between P and Q is 135^0; so the angles between P and R, and Q and R are respectively 120^0 and 105^0.

11. Let C be the ring at the end of BC, ACW the string which, passing through C, supports the weight W, (fig. 35).

The ring being smooth the tension of ACW is the same on each side of C,

$$\therefore \angle ACB = BCW = 90^0 + ABC;$$

$$\therefore \text{ if } ABC = \theta,\ AB = a,\ BC = b,$$

$$\frac{b}{a} = \frac{\sin\{180^0 - \theta - (90^0 + \theta)\}}{\sin(90^0 + \theta)} = \frac{\cos 2\theta}{\cos\theta};$$

$$\therefore \cos^2\theta - \frac{b}{2a}\cos\theta = \tfrac{1}{2} \text{ or } \cos\theta = \frac{1}{4a}\{b + \sqrt{8a^2 + b^2}\}.$$

The radical is taken with a positive sign because θ must be $< 90^0$.

12. Resolving in directions of 1 and 3, (fig. 36)

$$X = \text{force in direction of } 1 = 1 + 2 \times \tfrac{1}{2} - 4 \times \frac{\sqrt{3}}{2} = 2 - 2\sqrt{3}$$

$$Y = \ldots\ldots\ldots\ldots\ldots\ldots 3 = 3 + 2 \times \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2} = 5 + \sqrt{3};$$

$$\therefore R^2 = X^2 + Y^2 = 44 + 2\sqrt{3},$$

and if θ be the inclination of the resultant to the direction of 1,

$$\tan\theta = \frac{Y}{X} = -\tfrac{1}{4}(5 + \sqrt{3})(\sqrt{3} + 1) = -\tfrac{1}{2}(4 + 3\sqrt{3}).$$

13. The tension of each string must be equal and opposite to the resultant of the tensions of the other two:

Let $\theta_1\, \theta_2\, \theta_3$ be the angles between 5 and 6, 4 and 5, 4 and 6 respectively; then (Art. 10, Ex. 3)

$$\cos\theta_1 = \frac{4^2 - 5^2 - 6^2}{2 \times 5 \times 6} = -\tfrac{3}{4} = \cos 138^0.\, 35';$$

$$\therefore \theta_1 = 138^0.\, 35';\ \text{so } \theta_2 = 97^0.\, 11',\ \theta_3 = 124^0.\, 14'.$$

14. The tension at each point of the string = either weight (P); Fig. (37)

$\therefore$ pressure on $A = 2P\cos 60^0 = P$, and acts vertically.

Pressure on B or $C = 2P\cos(\tfrac{1}{2} \times 120^0) = P$, and it acts in a direction making an angle of 60^0 with the vertical.

15. When three forces keep a point at rest, the sides of a triangle which are parallel to their directions will be proportional to them; consequently, if three forces, 3, 5, 9 could keep a point at rest, there would be a triangle whose sides are as 3, 5, and 9; which is impossible, since $3 + 5$ is < 9.